Clifford Algebra

A Computational Tool for Physicists

John Snygg

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A Computational Tool for Physicists

JOHN SNYGG

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To my wife, Virginia

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Since I had no co-author, I like to think that I wrote this book all by myself. However this book has benefited from considerable assistance from other people.

Early drafts of my manuscript became the object of substantial revisions because of suggestions I received. People who read significant portions of one draft or another and then made suggestions or merely gave encouragement were James Anderson, George Cvijanovich, Dennis DeTurck, William Heyl, Charles Kennel, Frank Morgan, Massimo Porrati, Engelbert Schucking, Kip Thorne, and Joanne Trimble.

I used a lot of suggestions but I also ignored a lot of advice so nobody other than myself can be blamed for any shortcomings in the book.

I relied a great deal on the creative resourcefulness of Ann Oster, Yoshiko Ishii, Elizabeth Rumics, and Isabell Woller who converted an underfunded library at Upsala College into a world-class research facility for my benefit. Many times I challenged them with inadequate citations of obscure publications. Invariably they somehow managed to fix up the citation and then obtain a copy of the sought-after paper through interlibrary loan.

The administration of Upsala College gave me access to one of the College's best PCs to do the drawings. Since I was a complete neophyte to the use of a PC, I was dependent on the patience of the support staff for Adobe Illustrator and Linguist's Software, Inc. I was also rescued from my PC ignorance by Julio Izquierdo, Ken Johnson, and Zhong Zhang. Finally, I would like to mention that although I did the drawings myself, I frequently deferred to the artistic judgment of my son Spencer.

CONTENTS

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INTRODUCTION

Much of Clifford algebra is quite simple minded. If this fact were generally recognized, Clifford algebra would be more widely used as a computational tool.

Entire books have been written on the calculus of manifolds. However, one does not need to master the contents of such a book before one is able to use different coordinate systems to solve problems. Similarly much of the material usually covered in books on Clifford algebra is unnecessary for a very broad spectrum of applications.

The applications discussed in this book range from special relativity and the rotating top at one end of this spectrum to general relativity and Dirac's equation for the electron at the other end. In Chapter 9, we present an elementary derivation of the Kerr metric which is the basis for the mathematics of black holes. In Chapter 10, we present a second derivation of the Kerr metric. This second derivation is more sophisticated but it is also more straightforward than the derivation presented in Chapter 9.

The math prerequisites for this book are the usual undergraduate sequence of courses in calculus plus one course in linear algebra. The physics prerequisites for most of the book correspond to that covered by an undergraduate physics major. However, a few applications discussed in the book may require some physics usually covered in the first year of graduate school.

Clifford algebra has become a virtual necessity for the study of some areas of physics and its use is expanding in other areas. Some physicists have been using Clifford algebra without realizing it. In quantum electrodynamics, it has been discovered that many algebraic manipulations involving Dirac matrices can be carried out most efficiently without reference to any particular matrix representation. This is Clifford algebra.

In Klein–Kaluza theories and dimensional renormalization theories, one must deal with spaces of dimensions other than the four used in Dirac's equation. In such situations one can no longer use the usual 4×4 matrix representations of Dirac matrices and if one wishes to consider spaces of arbitrary dimension, one is forced to avoid reference to any particular representation. This requires Clifford algebra. Clifford algebra is now being applied in a very conscious and explicit manner in the formulation of

superstring theories. Thus Clifford algebra has become an indispensable tool for those at the cutting edge of theoretical investigations.

Vector calculus has long been regarded as a universal language of physicists. In recent years there have been those who have strongly recommended that all serious physicists become conversant with differential forms. However Clifford algebra encompasses both of these areas of mathematics along with tensor calculus.

Another formalism that has been effectively used and promoted by Roger Penrose and others is that of spinors. It is interesting to note that in the appendix of their book, *Spinors and Space-Time*, Vol. 2 (1986, pp. 440–464), Penrose and Rindler use the structure of Clifford algebra to generalize the notion of spinors that has been used for 4-dimensional spaces with signature $(+, -, -, -)$ to spaces of arbitrary dimension.

The structure of differential forms and tangent vectors is embedded in the structure of Clifford algebra with only slight modifications. Thus advanced readers who already have some mastery of differential forms should find the content of much of this book to be familiar territory.

Actually, in the context of Clifford algebra, the formalism of differential forms and tangent vectors can be substantially simplified. In the usual formulation, $\mathbf{d}x^i$ and $\partial/\partial x^j$ are presented as coordinate bases of dual spaces. The distinction between these spaces is necessary when no metric is given. However, for most physical applications, one needs to introduce a nonsingular metric which generates an isomorphism between these spaces. Spaces that are isomorphic are essentially identical. In the formalism of differential forms as usually presented, one does not take advantage of this fact. Even in the presence of a metric, unnecessary distinctions are maintained. This creates a multitude of products, mappings, and spaces which require a considerable amount of bookkeeping in the notation.

In the usual formalism, the tangent vector $\partial/\partial x^j$ is treated as the image of the differential form $g_{ji}\,\mathbf{d}x^i$ under an isomorphic mapping. In the structure of Clifford algebra, the analogous relation becomes a simple equality: $\gamma_j = g_{ji}\gamma^i$. This simple difference in the two formalisms enables one to carry out computational manipulations in Clifford algebra which would either be awkward or illegal in the usual formalism of differential forms. All products that appear in Clifford algebra can be readily grasped by anyone familiar with matrix multiplication.

One important feature of this book is a substantial discussion of Fock–Ivanenko 2-vectors (Fock and Ivanenko 1929; Fock 1929). These 2-vectors were introduced by Vladimir Fock and Dimitrii Ivanenko to make Dirac's equation for the electron compatible with the dictates of general relativity. Using these 2-vectors, denoted by $\mathbf{\Gamma}_\alpha$, Dirac's equation becomes

$$\gamma^\alpha\left(\frac{\partial}{\partial x^\alpha} - \mathbf{\Gamma}_\alpha + \frac{ie}{\hbar c}A_\alpha\right)\psi = -\frac{imc}{\hbar}\psi.$$

In this context, the $\mathbf{\Gamma}_\alpha$'s may be regarded as components of a gauge field.

It has been said that gravitational fields are not Yang–Mills fields but it is interesting to note that

$$\frac{\partial}{\partial x^\alpha}\mathbf{\Gamma}_\beta - \frac{\partial}{\partial x^\beta}\mathbf{\Gamma}_\alpha - \mathbf{\Gamma}_\alpha\mathbf{\Gamma}_\beta + \mathbf{\Gamma}_\beta\mathbf{\Gamma}_\alpha = \tfrac{1}{2}\mathscr{R}_{\alpha\beta}$$

where $\mathscr{R}_{\alpha\beta}$ is the curvature 2-form. Furthermore, using the formalism of this text, a slightly generalized version of Einstein's field equations can be cast in the form of a Yang–Mills equation. Namely

$$\frac{1}{\sqrt{-g}}\nabla_\theta(\sqrt{-g}\tfrac{1}{2}\mathscr{R}^{\alpha\theta}) = \mathscr{I}^\alpha.$$

This is equivalent to

$$R^\alpha{}_{\eta,\nu}\gamma^\eta \wedge \gamma^\nu = 8\pi(T^\alpha{}_\eta - \tfrac{1}{2}T\delta^\alpha_\eta)_{;\nu}\gamma^\eta \wedge \gamma^\nu$$

where the $R^\alpha{}_\eta$'s are components of the Ricci tensor and the $T^\alpha{}_\eta$'s are components of the energy-momentum tensor. This form of Einstein's field equations admits the possibility of a nonzero cosmological constant.

I have found that the Fock–Ivanenko 2-vectors can be used to expedite the computation of both the Schwarzschild and the Kerr metric. These computations are carried out in this text.

Different readers of this book will have different interests. Some will be interested in some applications but not in others. Some readers may wish to read the mathematical portions and omit some or even most of the applications. This is generally possible. Several sections of the book are unnecessary for the understanding of the succeeding contents of the book. These sections have been designated with asterisks in the table of contents.

Clifford Algebra

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1

A TASTE OF CLIFFORD ALGEBRA IN EUCLIDEAN 3-SPACE

1.1 Reflections, Rotations, and Quaternions in E^3 via Clifford Algebra

One usually represents a vector $\boldsymbol{x}$ in a 3-dimensional Euclidean space E^3 by $\boldsymbol{x} = x^1\hat{\imath} + x^2\hat{\jmath} + x^3\hat{k}$ or $(x^1, x^2, x^3) = x^1(1, 0, 0) + x^2(0, 1, 0) + x^3(0, 0, 1)$. However there are many alternative representations that could be used. For example, we could write

$$\boldsymbol{x} = x^1\hat{\gamma}_1 + x^2\hat{\gamma}_2 + x^3\hat{\gamma}_3, \tag{1.1}$$

where

$$\hat{\gamma}_1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \qquad \hat{\gamma}_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix},$$

and

$$\hat{\gamma}_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}. \tag{1.2}$$

At first sight this may seem to be a pointless variation. However, representing a vector in terms of these square matrices enables us to multiply vectors in a way which would not otherwise be possible. We should first note that the matrices $\hat{\gamma}_1$, $\hat{\gamma}_2$, and $\hat{\gamma}_3$ have some special algebraic properties. In particular

$$(\hat{\gamma}_1)^2 = (\hat{\gamma}_2)^2 = (\hat{\gamma}_3)^2 = I, \tag{1.3}$$

where I is the identity matrix. Furthermore

$$\hat{\gamma}_2\hat{\gamma}_3 + \hat{\gamma}_3\hat{\gamma}_2 = \hat{\gamma}_3\hat{\gamma}_1 + \hat{\gamma}_1\hat{\gamma}_3 = \hat{\gamma}_1\hat{\gamma}_2 + \hat{\gamma}_2\hat{\gamma}_1 = 0. \tag{1.4}$$

A set of matrices that satisfy Eqs. (1.3) and (1.4) is said to form the basis for the *Clifford algebra* associated with Euclidean 3-space. There are matrices other than those presented in Eq. (1.2) which satisfy Eqs. (1.3) and (1.4). (See Problem 1.2.) In the formalism of Clifford algebra, one never deals with the components of any specific matrix representation. We have introduced the matrices of Eq. (1.2) only to demonstrate that there exist entities which satisfy Eqs. (1.3) and (1.4).

Now let us consider the product of two vectors. Suppose $\boldsymbol{y} = y^1\hat{\gamma}_1 + y^2\hat{\gamma}_2 + y^3\hat{\gamma}_3$. Then

$$\begin{aligned}\boldsymbol{xy} = {} & (x^1y^1 + x^2y^2 + x^3y^3)I + x^2y^3\hat{\gamma}_2\hat{\gamma}_3 + x^3y^2\hat{\gamma}_3\hat{\gamma}_2 \\ & + x^3y^1\hat{\gamma}_3\hat{\gamma}_1 + x^1y^3\hat{\gamma}_1\hat{\gamma}_3 + x^1y^2\hat{\gamma}_1\hat{\gamma}_2 + x^2y^1\hat{\gamma}_2\hat{\gamma}_1.\end{aligned} \tag{1.5}$$

Using the relations of Eq. (1.4), we have:

$$\begin{aligned}\boldsymbol{xy} = {} & (x^1y^1 + x^2y^2 + x^3y^3)I + (x^2y^3 - x^3y^2)\hat{\gamma}_2\hat{\gamma}_3 \\ & + (x^3y^1 - x^1y^3)\hat{\gamma}_3\hat{\gamma}_1 + (x^1y^2 - x^2y^1)\hat{\gamma}_1\hat{\gamma}_2.\end{aligned} \tag{1.6}$$

(Note: $\boldsymbol{xy} \neq \boldsymbol{yx}$.)

We notice that the coefficient of I in Eq. (1.6) is the usual scalar product $\langle \boldsymbol{x}, \boldsymbol{y}\rangle$. Furthermore the coefficients of $\hat{\gamma}_2\hat{\gamma}_3$, $\hat{\gamma}_1\hat{\gamma}_3$, and $\hat{\gamma}_1\hat{\gamma}_2$ are the three components of the cross product $\boldsymbol{x} \times \boldsymbol{y}$.

By considering all possible products, one obtains an 8-dimensional space spanned by $\{I, \hat{\gamma}_1, \hat{\gamma}_2, \hat{\gamma}_3, \hat{\gamma}_2\hat{\gamma}_3, \hat{\gamma}_3\hat{\gamma}_1, \hat{\gamma}_1\hat{\gamma}_2, \hat{\gamma}_1\hat{\gamma}_2\hat{\gamma}_3\}$, where

$$\hat{\gamma}_2\hat{\gamma}_3 = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}, \quad \hat{\gamma}_3\hat{\gamma}_1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}$$

$$\hat{\gamma}_1\hat{\gamma}_2 = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad \text{and} \quad \hat{\gamma}_1\hat{\gamma}_2\hat{\gamma}_3 = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}.$$

One might think that one could obtain higher order products. However, any

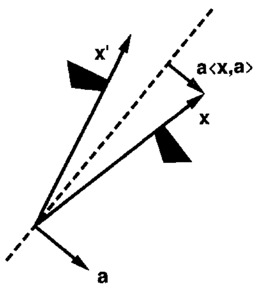


Fig. 1.1. Vector x' is the result of reflecting vector x with respect to the plane perpendicular to $\boldsymbol{a}$.

such higher order product will collapse to a multiple of one of the eight matrices already listed. For example:

$$\hat{\gamma}_1\hat{\gamma}_2\hat{\gamma}_3\hat{\gamma}_2 = \hat{\gamma}_1\hat{\gamma}_2(\hat{\gamma}_3\hat{\gamma}_2) = -\hat{\gamma}_1\hat{\gamma}_2(\hat{\gamma}_2\hat{\gamma}_3) = -\hat{\gamma}_1(\hat{\gamma}_2\hat{\gamma}_2)\hat{\gamma}_3 = -\hat{\gamma}_1\hat{\gamma}_3 = \hat{\gamma}_3\hat{\gamma}_1.$$

In this fashion, we have obtained an 8-dimensional vector space that is closed under multiplication. A vector space that is closed under multiplication is called an *algebra.* An algebra which arises from a vector space with a scalar product in the same manner as this example does from E^3 is called a *Clifford algebra.* (We will give a more formal definition of a Clifford algebra in Chapter 3.)

The matrices $\hat{\gamma}_1$, $\hat{\gamma}_2$, and $\hat{\gamma}_3$ are known as *Dirac matrices.* Any linear combination of Dirac matrices is a *1-vector.* A linear combination of $\hat{\gamma}_2\hat{\gamma}_3$, $\hat{\gamma}_3\hat{\gamma}_1$, and $\hat{\gamma}_1\hat{\gamma}_2$ is a *2-vector.* In the same vein, a multiple of I is a *0-vector* and any multiple of $\hat{\gamma}_1\hat{\gamma}_2\hat{\gamma}_3$ is a *3-vector.*

A general linear combination of vectors of possibly differing type is a *Clifford number.*

It will be helpful to use an abbreviated notation for products of Dirac matrices. In particular let

$$\hat{\gamma}_2\hat{\gamma}_3 = \hat{\gamma}_{23}, \qquad \hat{\gamma}_3\hat{\gamma}_1 = \hat{\gamma}_{31}, \qquad \hat{\gamma}_1\hat{\gamma}_2 = \hat{\gamma}_{12}, \qquad \text{and} \qquad \hat{\gamma}_1\hat{\gamma}_2\hat{\gamma}_3 = \hat{\gamma}_{123}.$$

The algebraic properties of Clifford numbers provide us with a convenient way of representing reflections and rotations. Suppose $\boldsymbol{a}$ is a vector of unit length perpendicular to a plane passing through the origin and $\boldsymbol{x}$ is an arbitrary vector in E^3 (see Fig. 1.1). In addition, suppose $\boldsymbol{x}'$ is the vector obtained from $\boldsymbol{x}$ by the reflection of $\boldsymbol{x}$ with respect to the plane corresponding to $\boldsymbol{a}$. Then

$$\boldsymbol{x}' = \boldsymbol{x} - 2\langle \boldsymbol{a}, \boldsymbol{x}\rangle \boldsymbol{a}. \tag{1.7}$$

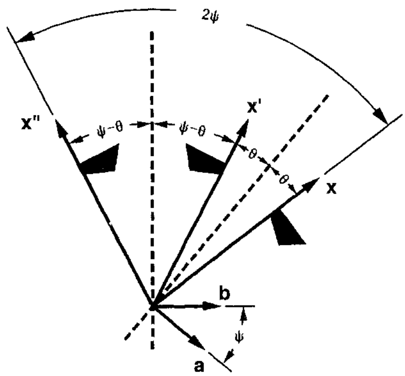


Fig. 1.2. When x is subjected to two successive reflections first with respect to a plane perpendicular to $\boldsymbol{a}$ and then with respect to a plane perpendicular to $\boldsymbol{b}$, the result is a rotation of x about an axis in the direction $\boldsymbol{a} \times \boldsymbol{b}$. The angle of rotation is twice the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$

From Eq. (1.6), it is clear that

$$\boldsymbol{ax} + \boldsymbol{xa} = 2\langle \boldsymbol{a}, \boldsymbol{x}\rangle I. \tag{1.8}$$

Using this relation, Eq. (1.7) becomes

$$\boldsymbol{x}' = \boldsymbol{x} - \boldsymbol{axa} - \boldsymbol{x}(\boldsymbol{a})^2. \tag{1.9}$$

However, using Eq. (1.6) again and the fact that $\boldsymbol{a}$ is a vector of unit length, we have

$$(\boldsymbol{a})^2 = \langle \boldsymbol{a}, \boldsymbol{a}\rangle I = I.$$

Thus Eq. (1.9) becomes

$$\boldsymbol{x}' = -\boldsymbol{axa}. \tag{1.10}$$

A rotation is the result of two successive reflections. (See Fig. 1.2.)

From Fig. 1.2, it is clear that if $\boldsymbol{x}'' = -\boldsymbol{bx}'\boldsymbol{b} = \boldsymbol{baxab}$, then $\boldsymbol{x}''$ is the vector that results from rotating vector $\boldsymbol{x}$ through an angle 2ψ about an axis with the direction of the axial vector $\boldsymbol{a} \times \boldsymbol{b}$. We can rewrite this relation in the form:

$$\boldsymbol{x}'' = \mathscr{R}\boldsymbol{x}\mathscr{R}^{-1} \qquad \text{where} \qquad \mathscr{R} = \boldsymbol{ba}. \tag{1.11}$$

It is useful to explicitly compute the product $\boldsymbol{ba}$ and interpret the separate components. If

$$\boldsymbol{a} = a^1\hat{\gamma}_1 + a^2\hat{\gamma}_2 + a^3\hat{\gamma}_3$$

and

$$\boldsymbol{b} = b^1\hat{\gamma}_1 + b^2\hat{\gamma}_2 + b^3\hat{\gamma}_3,$$

then from Eq. (1.6)

$$\mathscr{R} = \boldsymbol{ba} = \langle \boldsymbol{a}, \boldsymbol{b}\rangle I - (\boldsymbol{a} \times \boldsymbol{b})^1\hat{\gamma}_{23} - (\boldsymbol{a} \times \boldsymbol{b})^2\hat{\gamma}_{31} - (\boldsymbol{a} \times \boldsymbol{b})^3\hat{\gamma}_{12}.$$

Since both $\boldsymbol{a}$ and $\boldsymbol{b}$ are vectors of unit length $\langle \boldsymbol{a}, \boldsymbol{b}\rangle = \cos\psi$. Furthermore the magnitude of $\boldsymbol{a} \times \boldsymbol{b}$ is $\sin\psi$. For this reason, in the usual row matrix representation:

$$\boldsymbol{a} \times \boldsymbol{b} = ((\boldsymbol{a} \times \boldsymbol{b})^1, (\boldsymbol{a} \times b)^2, (\boldsymbol{a} \times \boldsymbol{b})^3) = \sin\psi(n^1, n^2, n^3)$$

where b^1, n^2, and n^3 are the direction cosines of the axial vector $\boldsymbol{a} \times \boldsymbol{b}$. With this thought in mind, we have

$$\mathscr{R} = I\cos\psi - \sin\psi(n^1\hat{\gamma}_{23} + n^2\hat{\gamma}_{31} + n^3\hat{\gamma}_{12}).$$

We should note that ψ represents $\frac{1}{2}$ the angle of rotation. If θ is the actual angle of rotation, we then have

$$\mathscr{R} = I\cos\frac{\theta}{2} - \sin\frac{\theta}{2}(n^1\hat{\gamma}_{23} + n^2\hat{\gamma}_{31} + n^3\hat{\gamma}_{12}). \tag{1.12}$$

To obtain $\mathscr{R}^{-1}$ from $\mathscr{R}$, one can replace θ by $-\theta$ or reverse the order of the Dirac matrices. In either case

$$\mathscr{R}^{-1} = I\cos\frac{\theta}{2} + \sin\frac{\theta}{2}(n^1\hat{\gamma}_{23} + n^2\hat{\gamma}_{31} + n^3\hat{\gamma}_{12}). \tag{1.13}$$

Readers who are familiar with quaternions should note the striking resemblance between the right-hand side of Eq. (1.12) and the rotation operator written in terms of quaternions. In the theory of quaternions introduced by William Hamilton in 1843, the same rotation operator written as a *quaternion* appears in the form:

$$\mathscr{R} = 1\cos\frac{\theta}{2} + \sin\frac{\theta}{2}(n^1\boldsymbol{i} + n^2\boldsymbol{j} + n^3\boldsymbol{k}). \tag{1.14}$$

Indeed, we can identify $\boldsymbol{i}$, $\boldsymbol{j}$, and $\boldsymbol{k}$ respectively with $-\gamma_{23}$, $-\gamma_{31}$, and $-\gamma_{12}$. The algebraic relations which define $\boldsymbol{i}$, $\boldsymbol{j}$, and $\boldsymbol{k}$ are

$$(\boldsymbol{i})^2 = (\boldsymbol{j})^2 = (\boldsymbol{k})^2 = -1, \tag{1.15a}$$

$$\boldsymbol{jk} = -\boldsymbol{kj} = \boldsymbol{i}, \qquad \boldsymbol{ki} = -\boldsymbol{ik} = \boldsymbol{j}, \tag{1.15b}$$

$$\boldsymbol{ij} = -\boldsymbol{ji} = \boldsymbol{k}. \tag{1.15c}$$

The same equations written in terms of the 2-vectors associated with E^3 become

$$(-\hat{\gamma}_{23})(-\hat{\gamma}_{23}) = (-\hat{\gamma}_{31})(-\hat{\gamma}_{31}) = (-\hat{\gamma}_{12})(-\hat{\gamma}_{12}) = -I, \tag{1.16a}$$

$$(-\hat{\gamma}_{31})(-\hat{\gamma}_{12}) = -(\hat{\gamma}_{12})(-\hat{\gamma}_{31}) = (-\hat{\gamma}_{23}), \tag{1.16b}$$

$$(-\hat{\gamma}_{12})(-\hat{\gamma}_{23}) = -(-\hat{\gamma}_{23})(-\hat{\gamma}_{12}) = (-\hat{\gamma}_{31}), \tag{1.16c}$$

$$(-\hat{\gamma}_{23})(-\hat{\gamma}_{31}) = -(-\hat{\gamma}_{31})(-\hat{\gamma}_{23}) = (-\hat{\gamma}_{12}), \tag{1.16d}$$

In Hamilton's formulation, a vector $\boldsymbol{x}$ is represented as $x^1\boldsymbol{i} + x^2\boldsymbol{j} + x^3\boldsymbol{k}$ and the rotated vector $\boldsymbol{x}''$ is computed by the quaternion version of Eq. (1.11). However the Hamilton approach fails to make the distinction between an ordinary vector and an *axial* or *pseudo-vector.*

An example of a pseudo-vector in E^3 is the cross product $\boldsymbol{a} \times \boldsymbol{b}$ formed from the composition of two ordinary vectors $\boldsymbol{a}$ and $\boldsymbol{b}$. In standard elementary treatments of vectors in E^3, no distinction is made between an ordinary vector and a pseudo-vector. This is largely due to the fact that a pseudo-vector in E^3 has the same number of components as an ordinary vector. Furthermore both behave in the same fashion under a rotation.

However, under a reflection, the two entities behave quite differently. For example suppose we consider a reflection with respect to the yz-plane. Let

$$\boldsymbol{a} = (a^1, a^2, a^3), \qquad \boldsymbol{b} = (b^1, b^2, b^3),$$

and

$$\boldsymbol{a} \times \boldsymbol{b} = (a^2b^3 - a^3b^2, a^3b^1 - a^1b^3, a^1b^2 - a^2b^1).$$

After reflection with respect to the yz-plane, the reflected vectors $\boldsymbol{a}'$ and $\boldsymbol{b}'$ may be obtained respectively from $\boldsymbol{a}$ and $\boldsymbol{b}$ by changing the sign of the first component in each vector. That is

$$\boldsymbol{a}' = (-a^1, a^2, a^3) \qquad \text{and} \qquad \boldsymbol{b}' = (-b^1, b^2, b^3).$$

On the other hand

$$\boldsymbol{a}' \times \boldsymbol{b}' = ((a^2b^3 - a^3b^2), -(a^3b^1 - a^1b^3), -(a^1b^2 - a^2b^1)).$$

Thus in contrast to the ordinary vectors $\boldsymbol{a}'$ and $\boldsymbol{b}'$, we see that it is the sign of the second and third components which have reversed sign for the pseudo-vector $\boldsymbol{a}' \times \boldsymbol{b}'$.

In the context of Clifford algebra, this distinction between ordinary vectors and axial or pseudo-vectors is automatic. In the formalism of Clifford algebra, ordinary vectors appear as 1-vectors and cross products appear as 2-vectors. A similar distinction is made between *scalars* such as $\langle \boldsymbol{a}, \boldsymbol{b} \rangle$ which

do not change sign under a reflection and *pseudo-scalars* such as $\langle \boldsymbol{a}, \boldsymbol{b} \times \boldsymbol{c} \rangle$ which do change sign under reflection. In the formalism of the Clifford algebra associated with E^3, scalars appear as 0-vectors and pseudo-scalars appear as 3-vectors.

Even in the context of Clifford algebra, some care must be taken in the computation of reflections. For example, suppose $\boldsymbol{a}$ and $\boldsymbol{b}$ are 1-vectors and $\boldsymbol{a}'$ and $\boldsymbol{b}'$ represent the reflections of $\boldsymbol{a}$ and $\boldsymbol{b}$ with respect to the yz-plane. Then

$$\boldsymbol{a}' = -\hat{\gamma}_1 \boldsymbol{a} \hat{\gamma}_1. \tag{1.17}$$

The cross product of $\boldsymbol{a}$ and $\boldsymbol{b}$ can be represented by the antisymmetric product

$$\tfrac{1}{2}(\boldsymbol{ab} - \boldsymbol{ba}) = (a^2b^3 - a^3b^2)\hat{\gamma}_{23} + (a^3b^1 - a^1b^3)\hat{\gamma}_{31} + (a^1b^2 - a^2b^1)\hat{\gamma}_{12}.$$

We observe that

$$\boldsymbol{a}'\boldsymbol{b}' = (-\hat{\gamma}_1 \boldsymbol{a} \hat{\gamma}_1)(-\hat{\gamma}_1 \boldsymbol{b} \hat{\gamma}_1) = \hat{\gamma}_1 \boldsymbol{ab} \hat{\gamma}_1.$$

Since a similar relation holds for the product $\boldsymbol{b}'\boldsymbol{a}'$, we have

$$\tfrac{1}{2}(\boldsymbol{a}'\boldsymbol{b}' - \boldsymbol{b}'\boldsymbol{a}') = \hat{\gamma}_1 \tfrac{1}{2}(\boldsymbol{ab} - \boldsymbol{ba})\hat{\gamma}_1. \tag{1.18}$$

We note that Eq. (1.8) has the same form as Eq. (1.7) except for a difference in sign.

The problem of the choice of sign that appears with reflections does not occur with rotations. This is true even in higher dimensional spaces where vectors of higher order can occur. Suppose $\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_p$ are all 1-vectors. If

$$\boldsymbol{a}'_k = \mathscr{R} \boldsymbol{a}_k \mathscr{R}^{-1} \quad \text{for } k = 1, 2, \ldots, p$$

then

$$\begin{aligned} \boldsymbol{a}'_1 \boldsymbol{a}'_2 \boldsymbol{a}'_3 \ldots \boldsymbol{a}'_p &= (\mathscr{R} \boldsymbol{a}_1 \mathscr{R}^{-1})(\mathscr{R} \boldsymbol{a}_2 \mathscr{R}^{-1}) \ldots (\mathscr{R} \boldsymbol{a}_p \mathscr{R}^{-1}) \\ &= \mathscr{R}(\boldsymbol{a}_1 \boldsymbol{a}_2 \boldsymbol{a}_3 \ldots \boldsymbol{a}_p)\mathscr{R}^{-1}. \end{aligned}$$

In closing this section, we should note that in the formalism of Clifford algebra, the rotation operators are double valued. That is if $\boldsymbol{x}' = \mathscr{R} \boldsymbol{x} \mathscr{R}^{-1}$ then $\mathscr{R}$ can be replaced by $-\mathscr{R}$. This point will be pursued a little further in the next section.

It should also be noted that the idea of decomposing a rotation into a product of reflections can be extended to n-dimensional Euclidean or pseudo-Euclidean spaces. For such spaces it can be shown that any rotation is the product of an even number $\leqslant n$ of reflections. This theorem was first proved by Élie Cartan for both the real and the complex domains (1938, pp. 13–17; 1966, pp. 10–12). This result was later extended to vector spaces over scalar fields with characteristic $\neq 2$ by Jean Dieudonné (1948, pp. 20–22).

Problem 1.1. From the form of Eq. (1.11), it is clear that if the rotation operators $\mathscr{R}$ and $\mathscr{R}'$ represent two successive rotations, then the combined rotation is presented by the product $\mathscr{R}'\mathscr{R}$. Use this fact and Eq. (1.12) to show that a 90° rotation about the x-axis followed by a 90° rotation about the y-axis is equivalent to a 120° rotation about the axis which has the direction of the vector $(1, \quad 1, \quad -1)$.

Problem 1.2. There are many representations that can be used for $\hat{\gamma}_1$, $\hat{\gamma}_2$, and $\hat{\gamma}_3$. One convenient representation is that using *Pauli matrices* σ_1, σ_2, and σ_3. That is, we can let

$$\hat{\gamma}_1 = \sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad \hat{\gamma}_2 = \sigma_2 = \begin{bmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{bmatrix},$$

and

$$\hat{\gamma}_3 = \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Show that in this representation Eq. (1.3) and Eq. (1.4) are satisfied.

Problem 1.3. In this representation introduced in Problem 1.2, the quaternions $\boldsymbol{i}, \boldsymbol{j}$, and $\boldsymbol{k}$ are represented by complex 2×2 matrices. In particular

$$\boldsymbol{i} = -\hat{\gamma}_{23} = -\mathrm{i}\sigma_1 = \begin{bmatrix} 0 & -\mathrm{i} \\ -\mathrm{i} & 0 \end{bmatrix},$$

$$\boldsymbol{j} = -\hat{\gamma}_{31} = -\mathrm{i}\sigma_2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$

and

$$\boldsymbol{k} = -\hat{\gamma}_{12} = -\mathrm{i}\sigma_3 = \begin{bmatrix} -\mathrm{i} & 0 \\ 0 & \mathrm{i} \end{bmatrix}.$$

In this representation the rotation operator

$$\mathscr{R} = I\cos\frac{\theta}{2} + \sin\frac{\theta}{2}(\boldsymbol{i}n^1 + \boldsymbol{j}n^2 + \boldsymbol{k}n^3)$$

$$= \begin{bmatrix} \cos\frac{\theta}{2} - \mathrm{i}n^3\sin\frac{\theta}{2} & -\sin\frac{\theta}{2}(n^2 + \mathrm{i}n^1) \\ \sin\frac{\theta}{2}(n^2 - \mathrm{i}n^1) & \cos\frac{\theta}{2} + \mathrm{i}n^3\sin\frac{\theta}{2} \end{bmatrix}$$

Show that, in this representation, the matrix representing $\mathscr{R}$ is unitary and has determinant equal to 1. (From this result, it is clear that the algebraic

properties of the double-valued rotation operators can be ascertained by studying the algebraic properties of 2×2 unitary matrices whose determinant is 1. For this reason the group of double-valued rotation operators is labeled SU(2). The letter U indicates "unitary". The letter S indicates "special" which in the context of group representation theory means the determinant equals 1.)

Problem 1.4. Suppose

$$\mathscr{R} = I \cos\frac{\theta}{2} + \hat{\boldsymbol{n}} \sin\frac{\theta}{2},$$

where

$$\hat{\boldsymbol{n}} = n^1\boldsymbol{i} + n^2\boldsymbol{j} + n^3\boldsymbol{k} = -n^1\hat{\gamma}_{23} - n^2\hat{\gamma}_{31} - n^3\hat{\gamma}_{12}.$$

Show that $\exp[\hat{n}(\theta/2)] = \mathscr{R}$. Hint: represent $\exp(\hat{\boldsymbol{n}}[(\theta/2)]$ by a Taylor's series and then separate the odd and even powers of $\hat{\boldsymbol{n}}$.

1.2 The 4π Periodicity of the Rotation Operator

From the consequences of the last section, we see that if the vector $\boldsymbol{x}(\theta)$ represents the result of rotating vector $\boldsymbol{x}(0)$ through an angle θ, then we can represent the rotation in the form:

$$\boldsymbol{x}(\theta) = \mathscr{R}(\theta)\boldsymbol{x}(0)\mathscr{R}^{-1}(\theta) \tag{1.19}$$

where

$$\mathscr{R}(\theta) = I \cos\frac{\theta}{2} - \hat{\boldsymbol{n}} \sin\frac{\theta}{2},$$

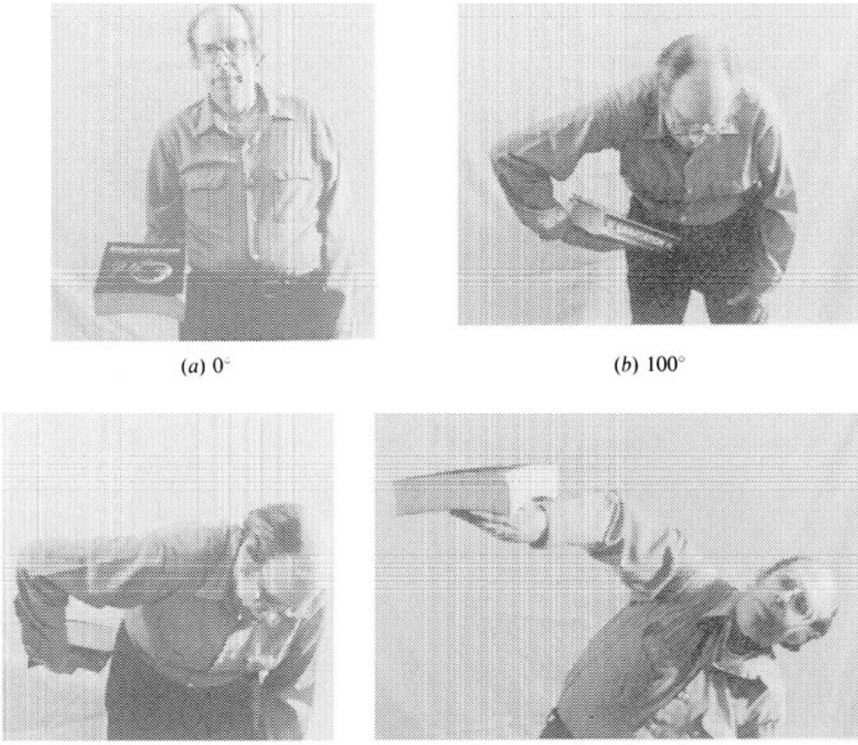
$$\hat{\boldsymbol{n}} = n^1\hat{\gamma}_{23} + n^2\hat{\gamma}_{31} + n^3\hat{\gamma}_{12},$$

and n^1, n^2, along with n^3, are the direction cosines of the axis of rotation.

Although $\boldsymbol{x}(\theta)$ has a period of 2π, $\mathscr{R}(\theta)$ has a period of 4π! With the development of quantum mechanics in the 1920s, it became recognized that 4π periodicities do occur in nature. To explain the observed structure of the hydrogen energy spectrum, it was necessary to attribute to the electron a spin of $\frac{1}{2}$ and a periodicity of 4π. In recent years, it has become more widely recognized that objects larger than electrons also have 4π periodicities (Bolker 1973). A demonstration of this fact has been put forward by Edgar Riefin (1979).

For an object to display a 4π periodicity it is necessary that it be in some sense loosely attached to its surroundings.

To illustrate this, you may wish to carry out a demonstration. First hold a glass of water in the palm of your hand. The hand holding the glass may be left or right but it is important that your hand be under the glass with



(*a*) 0° (*b*) 100°

(*c*) 180° (*d*) 250°

Fig. 1.3. (*a*)–(*d*).

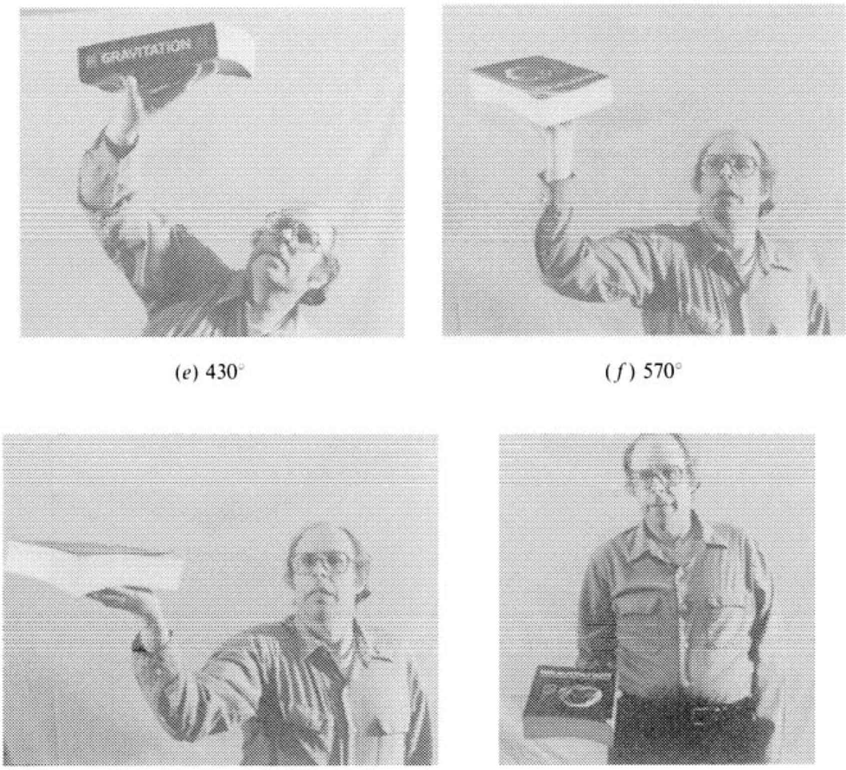
palm up. Then maintain a firm grip on the glass and rotate it 360° without moving your feet or spilling any water. You may do this in either the clockwise or counterclockwise direction. When you have completed this maneuver, you will find yourself in an awkward position with the glass slightly above your head and your elbow pointed upward. Clearly the relationship of the glass to you is quite different from what it was in its initial position. However if you continue the rotation, you may be surprised to find that your arm will unwind itself and the glass will return to its initial position with its initial relationship to you.

Thus the glass attached to your arm does not have a 2π periodicity but it does have a 4π periodicity.

This demonstration is shown in Fig. 1.3 where a book is used in place of a glass of water.

1.3 The Spinning Top (One Point Fixed)—Without Euler Angles

(The reader may omit this section without sacrificing his or her ability to comprehend the rest of this book.)



(*e*) 430° (*f*) 570°

(*g*) 650° (*h*) 720°

Fig. 1.3. (*e*)–(*h*) A book which does not have a 2π periodicity but does have a 4π periodicity.

One useful application of Clifford algebra is the spinning top problem with one point fixed. One of the most significant difficulties with this problem is that of parameterizing the rotational motion. It is somewhat awkward to write down the matrix representing a given rotation unless the rotation happens to be around the x, y, or z axis. On the other hand, any rotation in Euclidean 3-space can be decomposed into a succession of three rotations about coordinate axes. It is for this reason that the required rotational transformations for the spinning top are usually expressed in terms of Euler angles.

A drawback to the use of Euler angles is the fact that not all the Euler angles are intrinsic to the motion of the top. Despite claims to the contrary, none of the Euler angles corresponds to the spin of the top about its own axis. Using Clifford algebra, it is easy to represent a rotation about an arbitrary axis. This enables one to introduce a more intrinsic set of coordinates.

To discuss the problem of the motion of a rigid body about a fixed point, one must not only be able to express the coordinates of a point of the

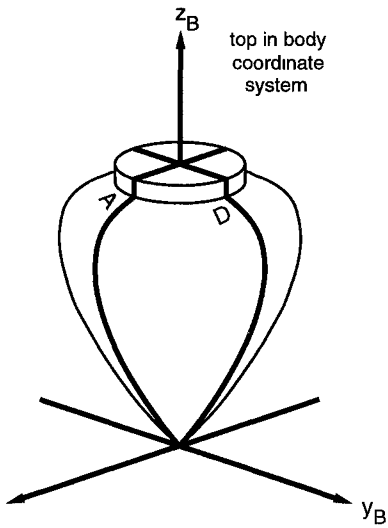


Fig. 1.4. The top in its own coordinate system.

body in the spatial coordinates of the observer, but also in the body coordinate system which is attached to the rigid body. This can be done conveniently using the formalism of Clifford algebra.

If the bottom tip of the top's axis of symmetry is stationary and coincident with the origin, it is possible to describe the position and orientation of the top at any instant in the spatial coordinate system of the observer in terms of three angles. I will refer to the magnitude of the spin of the top about its own axis as the *spin angle* and designate that angle by $\psi(t)$. (See Figs. 1.4 and 1.5.) I will refer to the angle between the vertical z-axis and the top's axis of symmetry as the *tilt angle* and designate that angle by $\theta(t)$. (See Fig. 1.6.) Finally, I designate the *angle of precession* by $\phi(t)$. The angle of precession is understood to be the angle between the xz-plane and the plane spanned by the top's axis and the z-axis. (See Fig. 1.6.)

In the body coordinate system used here, the top is stationary and the axis of symmetry is coincident with the z-axis and with the bottom tip of the top coincident with the origin.

To transform from the body coordinate system to the spatial coordinate system, we need the operator which will physically rotate the top from this stationary vertical position to the position and orientation of the top at some instant of time as seen by an observer. This rotation is somewhat complex but it may be considered to be the composition of two fairly simple rotations. The first is the rotation of the top by the spin angle $\psi(t)$ about the z-axis. The second is the rotation by the tilt angle $\theta(t)$ in the plane spanned by the z-axis and the axis of the top in the position as seen by the observer. The axis of rotation for this second rotation lies in the xy-plane with direction cosines $(-\sin\phi, \cos\phi, 0)$. (See Fig. 1.6.)

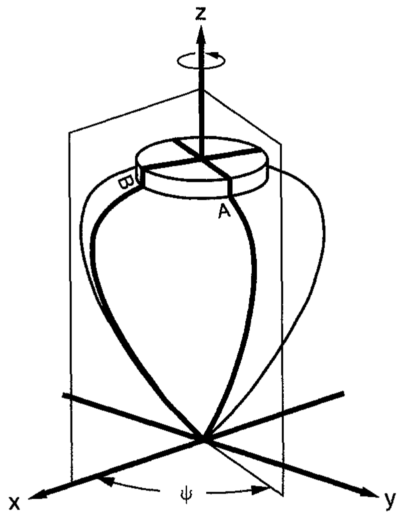


Fig. 1.5. The top rotated by angle ψ about its own axis.

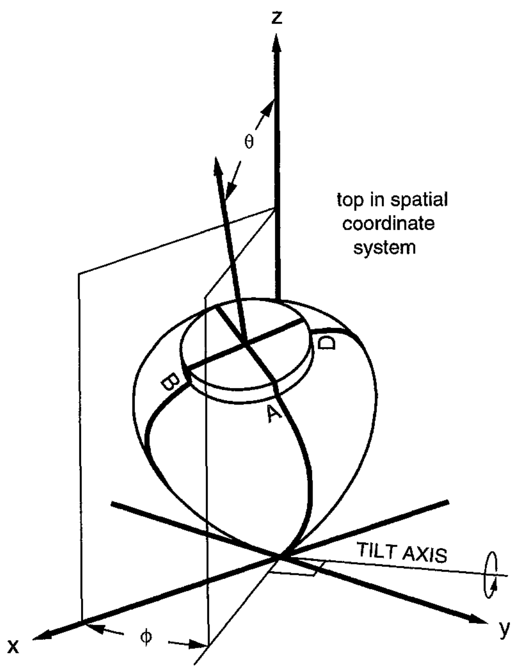


Fig. 1.6. The top tilted by angle θ from the z-axis. The plane containing the z-axis and the axis of the top makes an angle with the xz-plane That angle is designated by ϕ and denotes the angle of precession.

The first rotation operator is simply

$$\mathscr{R}_{\text{SPIN}} = I \cos\frac{\psi}{2} - \hat{\gamma}_{12} \sin\frac{\psi}{2}. \tag{1.20}$$

The second rotation operator is

$$\mathscr{R}_{\text{TILT}} = I \cos\frac{\theta}{2} + \hat{\gamma}_{23} \sin\frac{\theta}{2} \sin\phi - \hat{\gamma}_{31} \sin\frac{\theta}{2} \cos\phi. \tag{1.21}$$

If we designate an arbitrary point in the top as seen in the spatial coordinates of the observer by $\boldsymbol{x}$ and the same point in the body coordinate system by $\boldsymbol{x}_{\text{B}}$, we have

$$\boldsymbol{x} = \mathscr{R} x_{\text{B}} \mathscr{R}^{-1} \tag{1.22}$$

where

$$\mathscr{R} = \mathscr{R}_{\text{TILT}} \mathscr{R}_{\text{SPIN}}. \tag{1.23}$$

To obtain the equation of motion for a rotating top, we need to obtain expressions for both the kinetic energy T and the potential energy V. The kinetic energy of a rigid body is represented by the integral

$$T = \frac{1}{2} \int \rho(\boldsymbol{x})(\dot{\boldsymbol{x}})^2 \, \mathrm{d}V, \tag{1.24}$$

where $\rho(\boldsymbol{x})$ is the mass density at the point $\boldsymbol{x}$ and $\dot{\boldsymbol{x}}$ is the time derivative of $\boldsymbol{x}$.

To compute T, it is a virtual necessity to convert the variables of integration to the body coordinate system. With this thought in mind, let us now set out to compute $(\dot{\boldsymbol{x}})^2$ in the body coordinate system. From Eq. (1.22), we have

$$\dot{\boldsymbol{x}} = \dot{\mathscr{R}} \boldsymbol{x}_{\text{B}} \mathscr{R}^{-1} + \mathscr{R} x_{\text{B}} \dot{\mathscr{R}}^{-1}$$

or

$$\dot{\boldsymbol{x}} = \mathscr{R}(\mathscr{R}^{-1} \dot{\mathscr{R}} \boldsymbol{x}_{\text{B}} + \boldsymbol{x}_{\text{B}} \dot{\mathscr{R}}^{-1} \mathscr{R}) \mathscr{R}^{-1}. \tag{1.25}$$

(Note: $\dot{\mathscr{R}}^{-1}$ is the time derivative of $\mathscr{R}^{-1}$ and *not* the inverse of $\dot{\mathscr{R}}$ which may not exist.)

Since $\mathscr{R}^{-1}\mathscr{R} = 1$, it follows that $\dot{\mathscr{R}}^{-1}\mathscr{R} + \mathscr{R}^{-1}\dot{\mathscr{R}} = 0$. Using this relation to eliminate $\dot{\mathscr{R}}^{-1}\mathscr{R}$ in Eq. (1.25), we have

$$\dot{\boldsymbol{x}} = \mathscr{R}(\mathscr{R}^{-1} \dot{\mathscr{R}} \boldsymbol{x}_{\text{B}} - \boldsymbol{x}_{\text{B}} \mathscr{R}^{-1} \dot{\mathscr{R}}) \mathscr{R}^{-1}$$

and

$$(\dot{x})^2 = \mathscr{R}(\mathscr{R}^{-1}\dot{\mathscr{R}}x_B - x_B\mathscr{R}^{-1}\dot{\mathscr{R}})^2\mathscr{R}^{-1}$$

or

$$\mathscr{R}^{-1}(\dot{x})^2\mathscr{R} = (\mathscr{R}^{-1}\dot{\mathscr{R}}x_B - x_B\mathscr{R}^{-1}\dot{\mathscr{R}})^2.$$

Since $(\dot{x})^2$ is a scalar, it commutes with $\mathscr{R}$, and this last equation becomes

$$(\dot{x})^2 = (\mathscr{R}^{-1}\dot{\mathscr{R}}x_B - x_B\mathscr{R}^{-1}\dot{\mathscr{R}})^2. \tag{1.26}$$

To continue our computation of $(\dot{x})^2$ in terms of the body coordinates, we are now faced with the task of computing $\mathscr{R}^{-1}\dot{\mathscr{R}}$. From Eq. (1.23),

$$\dot{\mathscr{R}} = \dot{\mathscr{R}}_{\text{TILT}}\mathscr{R}_{\text{SPIN}} + \mathscr{R}_{\text{TILT}}\dot{\mathscr{R}}_{\text{SPIN}}$$

and

$$\mathscr{R}^{-1} = \mathscr{R}_{\text{SPIN}}^{-1}\mathscr{R}_{\text{TILT}}^{-1}.$$

Therefore

$$\mathscr{R}^{-1}\dot{\mathscr{R}} = \mathscr{R}_{\text{SPIN}}^{-1}(\mathscr{R}_{\text{TILT}}^{-1}\dot{\mathscr{R}}_{\text{TILT}})\mathscr{R}_{\text{SPIN}} + \mathscr{R}_{\text{SPIN}}^{-1}\dot{\mathscr{R}}_{\text{SPIN}}.$$

Completing this grubby calculation, one gets

$$\mathscr{R}^{-1}\dot{\mathscr{R}} = -\tfrac{1}{2}(\hat{\gamma}_{23}\omega_1 + \hat{\gamma}_{31}\omega_2 + \hat{\gamma}_{12}\omega_3). \tag{1.27}$$

where

$$\omega_1 = -\dot{\phi}\sin\theta\cos(\psi - \phi) + \dot{\theta}\sin(\psi - \phi), \tag{1.28}$$

$$\omega_2 = \dot{\phi}\sin\theta\sin(\psi - \phi) + \dot{\theta}\cos(\psi - \phi), \tag{1.29}$$

$$\omega_3 = \dot{\psi} - \dot{\phi}(1 - \cos\theta). \tag{1.30}$$

From Eqs. (1.26) and (1.27), we get

$$(\dot{x})^2 = [(\omega_1)^2 + (\omega_2)^2 + (\omega_3)^2](x_B)^2 - \sum_{j,k}\omega_j\omega_k x_B^j x_B^k. \tag{1.31}$$

Substituting this result into Eq. (1.24) gives us

$$T = \frac{1}{2}\sum_{j,k} I^{jk}\omega_j\omega_k$$

where

$$I^{jj} = \int \rho(x_B)[(x_B)^2 - (x_B^j)^2]\,\mathrm{d}V$$

and

$$I^{jk} = -\int \rho(\boldsymbol{x}_{\mathrm{B}}) x_{\mathrm{B}}^j x_{\mathrm{B}}^k \, \mathrm{d}V \qquad \text{for } j \neq k.$$

The I^{jk}'s are constants known as *moment of inertial coefficients.* If the top is cylindrically symmetric, we can identify the axis of symmetry with the z or x^3 axis of the body coordinate system. Because of the cylindrical symmetry it is not difficult to show that $I^{jk} = 0$ if $j \neq k$. For example

$$\begin{aligned} I^{31} = I^{13} &= -\iiint \rho(\sqrt{x^2 + y^2}, z) xz \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z \\ &= -\iiint \rho(r, z) r^2 z \cos\theta \, \mathrm{d}r \, \mathrm{d}\theta \, \mathrm{d}z \\ &= -\int_0^{2\pi} \cos\theta \, \mathrm{d}\theta \iint \rho(r, z) r^2 z \, \mathrm{d}r \, \mathrm{d}z = 0. \end{aligned}$$

In a similar fashion, we can show that $I^{11} = I^{22}$. If we redesignate I^{11} and I^{22} by I_1 and I^{33} by I_3, we have

$$T = \tfrac{1}{2} I_1 [(\omega_1)^2 + (\omega_2)^2] + \tfrac{1}{2} I_3 (\omega_3)^2.$$

Using Eqs. (1.28), (1.29), and (1.30), this becomes

$$T = \tfrac{1}{2} I_1 [(\dot{\phi})^2 \sin^2\theta + (\dot{\theta})^2] + \tfrac{1}{2} I_3 [\dot{\psi} - \dot{\phi}(1 - \cos\theta)]^2. \tag{1.32}$$

To obtain the equations of motion, we also need an expression for the potential energy V.

If we designate the total mass of the top by M and the distance of the center of mass from the fixed point along the axis of symmetry by h, then the potential energy is $V = Mgh\cos\theta$. (See Fig. 1.7.)

The Euler–Lagrange equations of motion are of the form

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial}{\partial \dot{\eta}} - \frac{\partial}{\partial \eta}\right)(T - V) = 0 \quad \text{where } \eta = \psi, \phi, \text{ or } \theta.$$

The first two of these equations are

$$\frac{\mathrm{d}}{\mathrm{d}t} I_3 [\dot{\psi} - \dot{\phi}(1 - \cos\theta)] = \frac{\mathrm{d}}{\mathrm{d}t}(I_3 \omega_3) = 0, \tag{1.33}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} I_1 (\dot{\phi} \sin^2\theta) - \frac{\mathrm{d}}{\mathrm{d}t} I_3 ([\dot{\psi} - \dot{\phi}(1 - \cos\theta)][1 - \cos\theta]) = 0. \tag{1.34}$$

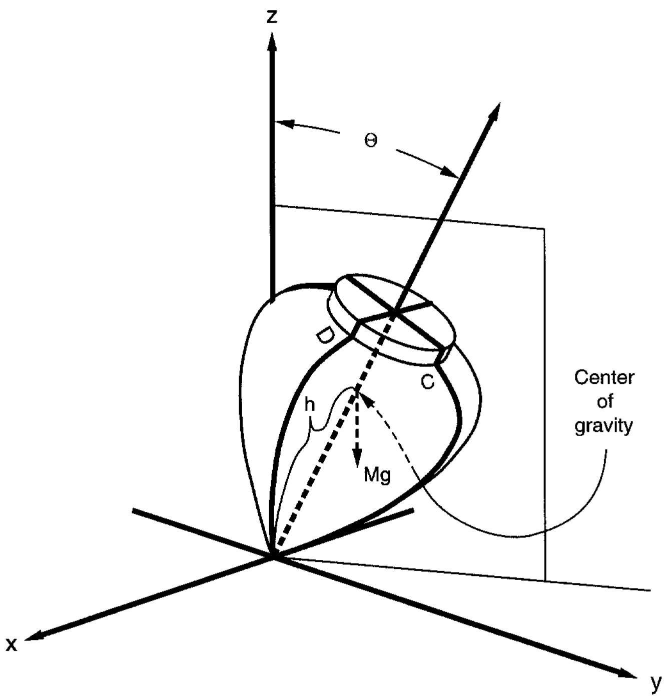


Fig. 1.7. The height of the center of gravity is equal to $h \times \cos\theta$.

From Eqs. (1.33) and (1.34), we get two constants of motion:

$$p_\psi = I_3[\dot\psi - \dot\phi(1 - \cos\theta)] = I_3\omega_3, \tag{1.35}$$

$$p_\phi = I_1\dot\phi \sin^2\theta - I_3\omega_3(1 - \cos\theta). \tag{1.36}$$

Furthermore the total energy is constant:

$$E = \tfrac{1}{2}I_1[(\dot\phi)^2 \sin^2\theta + (\dot\theta)^2] + \tfrac{1}{2}I_3(\omega_3)^2 + Mgh\cos\theta. \tag{1.37}$$

If we use Eq. (1.36) to eliminate $\dot\phi$ from Eq. (1.37), we get

$$\tfrac{1}{2}I_1(\dot\theta)^2 = (E - \tfrac{1}{2}I_3(\omega_3)^2) - \frac{[p_\phi + I_3\omega_3(1 - \cos\theta)]^2}{2I_1\sin^2\theta} - Mgh\cos\theta.$$

Multiplying this last equation by $\sin^2\theta$ or $1 - \cos^2\theta$, we have

$$\begin{aligned}\tfrac{1}{2}I_1(\sin\theta\dot\theta)^2 = {} & [E - \tfrac{1}{2}I_3(\omega_3)^2](1 - \cos^2\theta)\\ & - [p_\phi + I_3\omega_3(1 - \cos\theta)]^2/(2I_1)\\ & - Mgh\cos\theta(1 - \cos^2\theta).\end{aligned}$$

If we let $u = \cos\theta$, we then have $\dot{u} = -(\sin\theta)\dot{\theta}$ and

$$(I_1)^2(\dot{u})^2 = f(u), \tag{1.38}$$

where

$$f(u) = I_1[2E - I_3(\omega_3)^2 - 2Mghu](1 - u^2) - [p_\phi + I_3\omega_3 - I_3\omega_3 u]^2. \tag{1.39}$$

In principle, Eq. (1.38) can be solved for u or $\cos\theta$ as a function of time. That result can be used in Eq. (1.36) to obtain a solution for $\phi(t)$. Finally one can then use Eq. (1.35) to get a solution for $\psi(t)$.

Let us discuss one example. In the case of a toy top, you generally hold the top in some fixed position while you pull a string wrapped around the axis. When the string is pulled away from the top, the top is released. In this case the initial values of $\dot{\theta}$ and $\dot{\phi}$ are zero. If we use a zero subscript to label the initial values of our variables then $\dot{\theta}_0 = \dot{u}_0 = \dot{\phi}_0 = 0$.

From Eqs. (1.37) and (1.36)

$$2E - I_3(\omega_3)^2 = 2Mghu_0 \qquad \text{and} \qquad p_\phi + I_3\omega_3 = I_3\omega_3 u_0.$$

Substituting these relations into Eq. (1.38) gives us

$$(I_1)^2(\dot{u})^2 = 2MgI_1h(u_0 - u)(1 - u^2) - (I_3\omega_3)^2(u_0 - u)^2.$$

If we let

$$2MgI_1h = \alpha(I_3\omega_3)^2. \tag{1.40}$$

we then have

$$(I_1)^2(\dot{u})^2 = (I_3\omega_3)^2(u_0 - u)[-\alpha u^2 + u - (u_0 - \alpha)]. \tag{1.41}$$

The roots of the last factor on the right-hand side of Eq. (1.41) are

$$u = \frac{1 \pm \sqrt{1 - 4\alpha(u_0 - \alpha)}}{2\alpha}.$$

If the top has a high rate of spin then

$$\alpha = 2Mgh/(I_3\omega_3)^2 \ll 1.$$

Using this approximation and then using the first few terms in the Taylor expansion of $(1 - 4\alpha(u_0 - \alpha))^{\frac{1}{2}}$, our roots become

$$u_1 = u_0 - \alpha(1 - (u_0)^2), \tag{1.42}$$

$$u_2 = 1/\alpha - u_0. \tag{1.43}$$

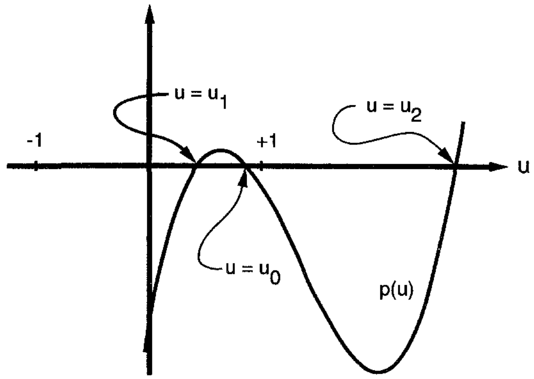


Fig. 1.8. The cubic polynomial $p(u) = (u_0 - u)[-\alpha u^2 + u - (u_0 - \alpha)] = (u_0 - u)(u - u_1) \times (u_2 - u)$, where $u = \cos\theta$, $u_1 \approx u_0 - \alpha[1 - (u_0)^2]$, $u_2 \approx 1/\alpha - u_0$, and $\alpha = 2Mgh/(I_3\omega_3)^2 \ll 1$.

With these approximations, we have

$$-\alpha u^2 + u - (u_0 - \alpha) = (u - u_1)(1 - \alpha(u + u_0)).$$

Equation (1.41) can then be rewritten as

$$\begin{aligned}(I_1)^2(\dot{u})^2 &= (I_3\omega_3)^2(u_0 - u)(u - u_1)(1 - \alpha(u + u_0)) \\ &= (I_3\omega_3)^2 p(u). \qquad (1.44)\end{aligned}$$

Since the left-hand side of Eq. (1.44) is nonnegative, physically meaningful values of u must be restricted to values for which the cubic polynomial $p(u)$ is positive. (See Fig. 1.8.) This means that

$$-1 < u_1 \leqslant u \leqslant u_0 \leqslant 1.$$

Recasting Eq. (1.44) as an integral, we have

$$t = \pm\frac{I_1}{I_3\omega_3}\int_{u_0}^{u}\frac{dx}{\sqrt{(u_0 - x)(x - u_1)(1 - \alpha(x + u_0))}}.$$

Initially u decreases as the upper tip of the axis dips. Thus for small values of t, we must choose the minus sign. After u attains its minimum value, it then begins to increase and we then have

$$t = \tfrac{1}{2}T + \frac{I_1}{I_3\omega_3}\int_{u_1}^{u}\frac{dx}{\sqrt{(u_0 - x)(x - u_1)(1 - \alpha(x + u_0))}}$$

where

$$\tfrac{1}{2}T = -\frac{I_1}{I_3\omega_3}\int_{u_0}^{u_1}\frac{\mathrm{d}x}{\sqrt{(u_0 - x)(x - u_1)(1 - \alpha(x + u_0))}}.$$

If we continue to use the high-spin approximation that $\alpha \ll 1$, then we can replace α by zero where it appears explicitly in the integral. Note: a non-zero value of α remains implicit in our approximation for u_1.

Using this approximation we can carry out the integration explicitly. One obtains

$$t = -\frac{I_1}{I_3\omega_3}\left[\text{arc sin}\left(\frac{2u - u_0 - u_1}{u_0 - u_1}\right) - \frac{\pi}{2}\right]$$

or

$$u = \cos\theta = \frac{u_0 + u_1}{2} + \frac{u_0 - u_1}{2}\cos\left(\frac{I_3\omega_3}{I_1}t\right). \tag{1.45}$$

This gives us the up and down motion of the upper tip of the axis. This motion is known as *nutation*. To obtain the rate of precession we note that from Eq. (1.36),

$$I_1\dot{\phi}(1 - u^2) + I_3\omega_3 u = I_3\omega_3 u_0$$

or

$$\dot{\phi} = \frac{I_3\omega_3}{I_1}\frac{(u_0 - u)}{1 - u^2}. \tag{1.46}$$

To get a more meaningful expression for $\dot{\phi}$, it is useful to use Eq. (1.42) to modify Eq. (1.45). The result is

$$u_0 - u = \frac{\alpha}{2}\,[1 - (u_0)^2]\left[1 - \cos\left(\frac{I_3\omega_3}{I_1}t\right)\right]. \tag{1.47}$$

In addition, it is not too difficult to show that

$$\frac{1 - (u_0)^2}{1 - u^2} = 1 + O(\alpha). \tag{1.48}$$

Incorporating Eqs. (1.47) and (1.48) into Eq. (1.46) and retaining only the lowest power of α, we get

$$\dot{\phi} = \frac{(I_3\omega_3)\alpha}{2I_1}\left[1 - \cos\left(\frac{I_3\omega_3}{I_1}t\right)\right].$$

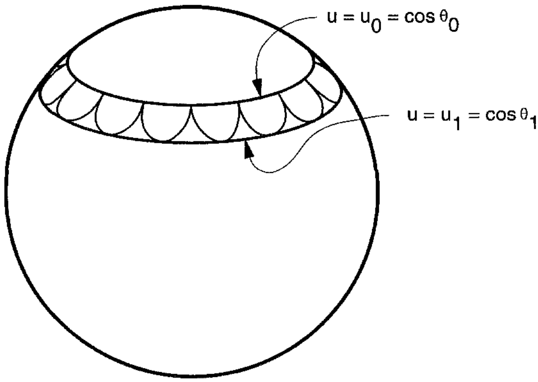


Fig. 1.9. Path of a point on the axis of rotation for a spinning top where the fixed point of the top is at the center of the sphere and the distance from the fixed point to the observed point is equal to the radius of the sphere.

Using Eq. (1.40), this becomes

$$\dot{\phi} = \frac{Mgh}{I_3\omega_3}\left[1 - \cos\left(\frac{I_3\omega_3}{I_1}t\right)\right]. \tag{1.49}$$

There are two things worth observing about Eq. (1.49). First: the angle ϕ is monotonic increasing or decreasing depending on the sign of ω_3. Furthermore $\dot{\phi} = 0$ when $t = 2\pi I_1 n/(I_3\omega_3)$ where n is any integer. This occurs each time $u = u_0$. The result is a hesitating jerky motion for the upper tip of the top. The path of the tip is illustrated in Fig. 1.9 where the lower tip of the top is at the center of the sphere. It should be noted that for a high rate of spin, the amplitude of the nutation is so small that visually it is observed only as a vibration.

Second: when ω_3 decreases due to friction, the average rate of procession increases!

A discussion of other solutions for the gyroscope appears in *Classical Mechanics* by Herbert Goldstein (1980). That text also contains a good bibliography.

Problem 1.5. Consider the rotation operator $\mathscr{R} = \mathscr{R}_{\mathrm{TILT}}\mathscr{R}_{\mathrm{SPIN}}$ defined by Eqs. (1.20), (1.21) and (1.23). Show that it can be computed alternatively by first carrying out the tilt and then rotating the top through the spin angle ψ about the axis with the direction ($\cos\phi\sin\theta$, $\sin\phi\sin\theta$, $\cos\theta$).

Problem 1.6. In elementary physics courses, one frequently sees an elementary solution for the rotating top derived from the simple notion that the rate of change for the angular momentum is equal to the torque due to gravity. This solution relies on a sometimes hidden assumption that the total

angular momentum has a direction coincident with the axis of symmetry for the top. This solution contains no nutation.

The condition for no nutation is that the cubic polynomial in Eq. (1.39) must contain a double root. That is equivalent to the requirement that $f(u_0) = 0$ and $f'(u_0) = 0$.

Show that if one eliminates E from those two equations and then uses Eq. (1.36) to eliminate p_ϕ, one arrives at the equation

$$I_1 u_0(\dot{\phi})^2 - I_3\omega_3\dot{\phi} + Mgh = 0. \tag{1.50}$$

Show that if one makes the high-spin approximation that

$$MgI_1h/(I_3\omega_3)^2 \ll 1,$$

then one of the roots for Eq. (1.50) is

$$\dot{\phi} = \frac{Mgh}{I_3\omega_3}.$$

Compare this result with Eq. (1.49). Compare this result with the solution that appears in more elementary texts.

2

A SAMPLE OF CLIFFORD ALGEBRA IN MINKOWSKI 4-SPACE

2.1 A Small Dose of Special Relativity

When the speed of sound is measured, it is found that the speed is independent of direction only if it is measured with respect to the air. If the air is moving at a rate of 20 kilometers per hour, an observer on the ground will discover that sound moving in the direction of the wind will move 20 kilometers per hour faster than it would when the air is still. Similarly sound moving against the wind will be slowed down.

During the nineteenth century, it was generally believed that light traveled through some kind of "ether" in much the same way as sound travels through air. In 1881, in an effort to measure the velocity of this ether with respect to earth, Albert Michelson designed an experiment which would compare the speed of light in different directions (1881). The result of the experiment was that the speed of light was the same in all directions. The experiment was refined and repeated six years later by Albert Michelson and Edward Morley, but the result was the same (1887).

It might be argued that at the time the measurements were taken, the earth was moving downstream in the same direction and speed as the ether. However, in the course of a year as the earth moves around the sun, the earth is hurtling through space in different directions and Michelson and Morley got the same null result at different times of the year.

To this day there are some holdouts who argue that the ether is dragged along by the earth. However, the overwhelming majority of physicists have abandoned the ether concept.

Aside from the result of Michelson and Morley, there were also some anomalies that appeared in the study of Maxwell's equations. According to Maxwell's equations, a magnetic field is generated by a moving charge. But what about an observer who moves with the same velocity as the charge? For such an observer, the charge is stationary. However, if the magnetic field exists, such an observer should be able to measure it.

To resolve this kind of paradox, some mathematically inclined physicists investigated the mathematical symmetries of Maxwell's equations. Hendrik

A. Lorentz (1904), Henri Poincaré (1905), and Albert Einstein (1905) published separate papers that presented a set of equations which have since become known as the Lorentz transformation.

It was Einstein who saw the physical consequence of these equations. For this reason, he is generally credited with the introduction of the special theory of relativity.

The Lorentz transformation can be derived from the assumption that the speed of light in a vacuum is a constant independent of direction for any observer who is moving at a constant speed relative to the source. Suppose we consider a spaceship flying at low level past a team of observers on top of Mt. Everest. Suppose further that the crew of the spaceship and the team of ground observers have previously agreed to set their clocks at $t = 0$ at the instant the spaceship swoops past the highest point of Mt. Everest. Also let us assume that at the instant that the time is zero in both the spaceship coordinate system and the earth coordinate system, a flashbulb is set off in an open window of the spaceship.

If the crew of the spaceship then monitors the position of the front edge of the expanding light wave, they will find that they are dealing with an expanding sphere whose radius is increasing at the speed of light c. Thus if (t, x^1, x^2, x^3) represents the coordinates of a point on the expanding sphere in a 4-dimensional space-time frame, then

$$(ct)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2 = 0. \tag{2.1}$$

For the team of observers on the ground who also monitor the wavefront, the result is the same. That is, in the coordinate system of the earth-bound observers, the wavefront is also a sphere expanding at the same speed. Thus if $(\bar{t}, \bar{x}^1, \bar{x}^2, \bar{x}^3)$ represents the coordinates of a point on the wavefront in their coordinate system, then

$$(c\bar{t})^2 - (\bar{x}^1)^2 - (\bar{x}^2)^2 - (\bar{x}^3)^2 = 0. \tag{2.2}$$

Suppose the orientation of the two coordinate systems agrees at time 0 and the spaceship is moving with velocity v along the x^1-axis. According to Newtonian mechanics, the relationship between the two coordinate systems is known as the Gallilean transformation:

$$\bar{t} = t, \tag{2.3a}$$

$$\bar{x}^1 = x^1 + vt, \tag{2.3b}$$

$$\bar{x}^2 = x^2, \tag{2.3c}$$

$$\bar{x}^3 = x^3. \tag{2.3d}$$

However, this system of equations is inconsistent with Eqs. (2.1) and (2.2).

In the first chapter we considered rotations in Euclidean 3-space. One feature of rotations is that they preserve the length of a vector. In some sense, we can introduce a generalized rotation in our space-time system which will do much the same thing. If $s = (t, x^1, x^2, x^3)$ represents a vector in Minkowski 4-space then it is useful to consider the "square of its length" to be

$$\langle s, s \rangle = (ct)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2. \tag{2.4}$$

Certainly if we could find a transformation which preserved such lengths, then Eq. (2.2) would follow from Eq. (2.1) and vice versa.

To retain the equivalence of Eqs. (2.1) and (2.2), it would not be necessary to restrict oneself to length-preserving transformations. One could combine a length-preserving transformation with the following:

$$\begin{aligned} \bar{t} &= kt, \\ \bar{x}^1 &= kx^1, \\ \bar{x}^2 &= kx^2, \\ \bar{x}^3 &= kx^3. \end{aligned}$$

However, according to Einstein, neither of the coordinate systems should be preferred and thus

$$t = k\bar{t} = (k)^2 t, \qquad x^1 = k\bar{x}^1 = (k)^2 x^1, \quad \text{etc.}$$

For this reason $k^2 = 1$, and $k = \pm 1$. However, for either value of k, the resulting transformation would be a length-preserving transformation.

Length-preserving transformations include reflections but if the speed of our spaceship is decreased to zero, our transformation should reduce to the identity transformation. This is not possible with a reflection or with the product of an odd number of reflections. Therefore, if we stick with linear transformations, we are left with the possibility of a generalized rotation. As we have already seen, the formalism of Clifford algebra is particularly useful to handle rotations. Suppose we represent a vector s in Minkowski 4-space by the Clifford number

$$s = \hat{\gamma}_0 ct + \hat{\gamma}_1 x^1 + \hat{\gamma}_2 x^2 + \hat{\gamma}_3 x^3. \tag{2.5}$$

If we construct a Clifford algebra analogous to that used for Euclidean 3-space, we would like to have $(s)^2$ correspond to $\langle s, s \rangle$. If we impose this requirement, then

$$(s)^2 = I\langle s, s \rangle = I[(ct)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2]. \tag{2.6}$$

However, for this equation to hold, we need a set of Dirac matrices with

somewhat different algebraic properties from those used for Euclidean 3-space. In particular if (2.6) is a consequence of Eq. (2.5) then

$$(\hat{\gamma}_0)^2 = -(\hat{\gamma}_1)^2 = -(\hat{\gamma}_2)^2 = -(\hat{\gamma}_3)^2 = I \tag{2.7a}$$

and

$$\hat{\gamma}_i\hat{\gamma}_j = -\hat{\gamma}_j\hat{\gamma}_i \quad \text{if } i \neq j. \tag{2.7b}$$

A standard set of matrices which satisfies these relations is

$$\hat{\gamma}_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \quad \hat{\gamma}_1 = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix},$$

$$\hat{\gamma}_2 = \begin{bmatrix} 0 & 0 & 0 & \mathrm{i} \\ 0 & 0 & -\mathrm{i} & 0 \\ 0 & -\mathrm{i} & 0 & 0 \\ \mathrm{i} & 0 & 0 & 0 \end{bmatrix}, \quad \hat{\gamma}_3 = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}. \tag{2.8}$$

An alternate representation can be obtained by multiplying each matrix $\hat{\gamma}_1$, $\hat{\gamma}_2$, and $\hat{\gamma}_3$ of Eq. (1.2) by $-\mathrm{i}$ and then relabeling the resulting matrices by $\hat{\gamma}_1$, $\hat{\gamma}_2$, and $\hat{\gamma}_3$. To complete the set, we can let

$$\hat{\gamma}_0 = \begin{bmatrix} 0 & 0 & 0 & \mathrm{i} \\ 0 & 0 & -\mathrm{i} & 0 \\ 0 & \mathrm{i} & 0 & 0 \\ -\mathrm{i} & 0 & 0 & 0 \end{bmatrix}.$$

Recognizing the relationship between the Dirac matrices of Euclidean 3-space and the three spatial Dirac matrices of the 4-dimensional Minkowski space-time geometry it is not too difficult to determine the form of the spatial rotation operators. Whereas in Euclidean 3-space

$$\mathscr{R} = I\cos\frac{\theta}{2} - \sin\frac{\theta}{2}(n^1\hat{\gamma}_{23} + n^2\hat{\gamma}_{31} + n^3\hat{\gamma}_{12}),$$

the same rotation operator in the 4-dimensional Minkowski space becomes

$$\mathscr{R} = I\cos\frac{\theta}{2} - \sin\frac{\theta}{2}(n^1(\mathrm{i}\hat{\gamma}_2)(\mathrm{i}\hat{\gamma}_3) + n^2(\mathrm{i}\hat{\gamma}_3)(\mathrm{i}\hat{\gamma}_1) + n^3(\mathrm{i}\hat{\gamma}_1)(\mathrm{i}\hat{\gamma}_2))$$

or

$$\mathscr{R} = I\cos\frac{\theta}{2} + \sin\frac{\theta}{2}(n^1\hat{\gamma}_{23} + n^2\hat{\gamma}_{31} + n^3\hat{\gamma}_{12}). \tag{2.9}$$

In view of the result of Problem 1.4, it is possible to write this same operator in the form

$$\mathscr{R} = \exp\left(\hat{n}\frac{\theta}{2}\right), \tag{2.10}$$

where $\hat{n} = n^1\hat{\gamma}_{23} + n^2\hat{\gamma}_{31} + n^3\hat{\gamma}_{12}$.

In Minkowski 4-space, one can also consider space-time rotations or *boosts*. For example, a rotation in the x^1t-plane can be represented by the operator

$$\mathscr{B} = \exp\left(\hat{\gamma}_{10}\frac{\phi}{2}\right). \tag{2.11}$$

It can be shown that an alternate representation for $\mathscr{B}$ is

$$\mathscr{B} = I\cosh\frac{\phi}{2} + \hat{\gamma}_{10}\sinh\frac{\phi}{2}. \tag{2.12}$$

(See Problem 2.1.)

The inverse of $\mathscr{B}$ may be computed by reversing the sign of ϕ or replacing $\hat{\gamma}_{10}$ by $\hat{\gamma}_{01}$. Suppose

$$s = \hat{\gamma}_0 ct + \hat{\gamma}_1 x^1 + \hat{\gamma}_2 x^2 + \hat{\gamma}_3 x^3$$

and

$$\bar{s} = \mathscr{B}s\mathscr{B}^{-1}. \tag{2.13}$$

Sorting out the separate components of $\bar{s}$, we have

$$c\bar{t} = ct\cosh\phi + x^1\sinh\phi, \tag{2.14a}$$

$$\bar{x}^1 = x^1\cosh\phi + ct\sinh\phi, \tag{2.14b}$$

$$\bar{x}^2 = x^2, \tag{2.14c}$$

$$\bar{x}^3 = x^3. \tag{2.14d}$$

This transformation applies not just to points on the expanding sphere of the wavefront but to any point. The coordinates of the open window of the spaceship in the spaceship system are $(t, x^1, x^2, x^3) = (t, 0, 0, 0)$. The same

point in the earth-bound system is $(\bar{t}, \bar{x}^1, \bar{x}^2, \bar{x}^3) = (\bar{t}, \bar{x}^1, 0, 0)$. But in the earth-bound system, the spaceship is observed to have a velocity v along the x^1-axis so

$$\bar{x}^1 = v\bar{t}. \tag{2.15}$$

Using these relations in Eqs. (2.14a, b) allows us to interpret the values of $\cosh\phi$ and $\sinh\phi$. In particular $c\bar{t} = ct\cosh\phi$ so $\bar{x}^1$, which equals $ct\sinh\phi$, must also equal $c\bar{t}(\sinh\phi/\cosh\phi)$.

Thus from Eq. (2.15),

$$v = c\,\frac{\sinh\phi}{\cosh\phi} \qquad \text{or} \qquad \sinh\phi = \frac{v}{c}\cosh\phi. \tag{2.16}$$

Since a general property of these hyperbolic functions is

$$\cosh^2\phi - \sinh^2\phi = 1,$$

it follows that

$$\cosh^2\phi[1 - (v/c)^2] = 1$$

and

$$\cosh\phi = \frac{1}{\sqrt{1-(v/c)^2}}. \tag{2.17}$$

From Eqs. (2.16) and (2.17), we also have

$$\sinh\phi = \frac{v/c}{\sqrt{1-(v/c)^2}}. \tag{2.18}$$

Using these relations, we have

$$c\bar{t} = \frac{ct + (x^1 v/c)}{\sqrt{1-(v/c)^2}}, \tag{2.19a}$$

$$\bar{x}^1 = \frac{x^1 + vt}{\sqrt{1-(v/c)^2}}, \tag{2.19b}$$

$$\bar{x}^2 = x^2, \tag{2.19c}$$

$$\bar{x}^3 = x^3. \tag{2.19d}$$

For an everyday mundane velocity v for which $v \ll c$, this system of equations reduces to the Gallilean transformation of Eqs. (2.3a–d).

Perhaps the biggest intellectual leap made by Einstein when he introduced his special theory of relativity was the idea that the elapsed time between two events will be different for different observers.

A result of this aspect of the special theory of relativity is that two velocities must be added in a somewhat peculiar way.

To carry out the composition of two boosts in the same direction it is useful to use the exponential representation for a boost operator:

$$\mathscr{B}(\phi) = I \cosh \frac{\phi}{2} + \hat{\gamma}_{10} \sinh \frac{\phi}{2} = \exp\left(\hat{\gamma}_{10} \frac{\phi}{2}\right). \tag{2.20}$$

If $\mathscr{B}(\phi_1)$ represents the transformation from the coordinate system of the first observer to the coordinate system of a second observer and $\mathscr{B}(\phi_2)$ represents the transformation from the system of the second observer to the system of the third observer, then $\mathscr{B}(\phi_2)\mathscr{B}(\phi_1)$ represents the transformation from the first observer to the third observer. However, from the exponential representation, it is clear that $\mathscr{B}(\phi_2)\mathscr{B}(\phi_1) = \mathscr{B}(\phi_2 + \phi_1)$. From Eq. (2.16), it follows that

$$\frac{v_1}{c} = \frac{\sinh \phi_1}{\cosh \phi_1} = \tanh \phi_1 \tag{2.21a}$$

where v_1 is the velocity of the first observer with respect to the second observer along the x^1-axis. In a similar fashion

$$\frac{v_2}{c} = \tanh(\phi_2), \tag{2.21b}$$

and

$$\frac{v}{c} = \tanh(\phi_2 + \phi_1), \tag{2.21c}$$

where v_2 is the velocity of the second observer with respect to the third observer and v is the velocity of the first observer with respect to the third observer. From the result of Problem 2.2, it follows that

$$\frac{v}{c} = \frac{\cosh(\phi_1 + \phi_2)}{\cosh(\phi_1 + \phi_2)} = \frac{\sinh \phi_1 \cosh \phi_2 + \cosh \phi_1 \sinh \phi_2}{\cosh \phi_1 \cosh \phi_2 + \sinh \phi_1 \sinh \phi_2}.$$

If we now divide both the numerator and the denominator of this last fraction by the product $\cosh \phi_1 \cosh \phi_2$, we get

$$\frac{v}{c} = \frac{\tanh \phi_1 + \tanh \phi_2}{1 + \tanh \phi_1 \tanh \phi_2}.$$

That is

$$\frac{v}{c} = \frac{v_1/c + v_2/c}{1 + (v_1 v_2/c^2)}$$

or

$$v = \frac{v_1 + v_2}{1 + (v_1 v_2/c^2)}. \tag{2.22}$$

In the world of Newtonian physics $v = v_1 + v_2$ and indeed in the physics of Einstein this is nearly true if $|v_1 v_2| \ll c^2$. A consequence of Eq. (2.22) is that one cannot attain speeds faster than light by adding two velocities less than the speed of light.

Suppose for example an observer on earth sees a glass spaceship pass by at a speed of $\frac{2}{3}$ the speed of light. Onboard the spaceship a passenger at the rear of the passenger compartment throws a frisbee forward at two-thirds the speed of light. According to Newtonian physics, the frisbee would be moving at the rate of $\frac{4}{3}$ the speed of light with respect to the observer on earth. However, according to the theory of special relativity, this speed would be

$$v = \frac{\frac{2}{3}c + \frac{2}{3}c}{1 + (\frac{2}{3})^2} = \frac{12}{13}c.$$

For another example, suppose that the passenger turned on a flashlight. Onboard the spaceship the speed of the light signal would be measured as c. If the speed of the spaceship were v, then the speed of the light signal measured by the earth-bound observer would be

$$\frac{v + c}{1 + \frac{v}{c}} = c!$$

Another interesting phenomenon that emerges from the special theory of relativity is that of *Lorentz contraction.* An object moving with uniform velocity v has its linear dimension in the direction of motion shortened by the factor of $\sqrt{1 - (v/c)^2}$. The dimensions perpendicular to the direction of motion are unaffected by the motion. To obtain this result requires some insight into the nature of measurement of length.

Consider the problem of measuring the length of a rod. Suppose in a coordinate frame which is at rest with respect to the rod, the space-time coordinates of the two ends at times t_1 and t_2 are $(t_1, 0, 0, 0)$ and $(t_2, L_o, 0, 0)$. Alternatively, we can replace the time coordinate t by ct. These same space-time points then become $(ct_1, 0, 0, 0)$ and $(ct_2, L_o, 0, 0)$. This alternate representation is somewhat more convenient and because of this convenience

we will use this alternate convention for representing points in Minkowski 4-space throughout the remainder of this chapter.

In the frame at rest with respect to the rod, one can determine the location of the two ends at possibly separate times t_1 and t_2 and then obtain the spatial distance L_o between the two points.

On the other hand, suppose we consider a second frame in which the rod is moving with velocity v in the x^1-direction. According to Eqs. (2.14a, b) the space-time locations of the two ends in this frame are

$$(ct_1 \cosh \phi, ct_1 \sinh \phi, 0, 0)$$

and

$$(ct_2 \cosh \phi + L_o \sinh \phi, L_o \cosh \phi + ct_2 \sinh \phi, 0, 0).$$

In this second frame, the distance between the ends of the rod is a meaningful quantity only when the locations of the two ends are measured at the same time. Suppose we determine the location of the two ends at time 0 as measured in the second frame. Then for the first point $t_1 = 0$, and the space-time coordinates for that point are (0, 0, 0, 0). For the second point, the situation is more complicated. For the second point, $ct_2 \cosh \phi + L_o \sinh \phi = 0$. Using this equation to eliminate t_2 from the non-zero spatial coordinate gives us the expression $(0, L_o \cosh \phi - L_o \sinh^2 \phi/\cosh \phi, 0, 0)$ for the second point. Thus the spatial distance between the two ends is

$$L = L_o(\cosh^2 \phi - \sinh^2 \phi)/\cosh \phi = L_o/\cosh \phi,$$

or

$$L = L_o\sqrt{1 - \left(\frac{v}{c}\right)^2}. \tag{2.23}$$

One aspect of special relativity which makes it fun is the many "paradoxes" that seem to arise. One "paradox" that arises from the Lorentz contraction is as follows: suppose we consider a rod which is somewhat longer than a vertical slot in a wall. According to the special theory of relativity, we should be able to give the rod sufficient speed in the vertical direction so as to shorten the rod down to that of the slot. If we then also added a small horizontal component to the velocity, the rod would pass through the slot. (See Figs. 2.1a, b.)

So far so good. But what happens in the rest frame of the rod? In the rest frame of the rod, the rod has its full rest frame length and the length of the slot is shorter. How would it then be possible for the rod to pass through the slot?

To understand the situation in the rest frame of the rod, compare Figs. 2.2a, b. For purposes of illustration, I have constructed the picture for the case when the relative velocity of the rod and wall is such that the Lorentz

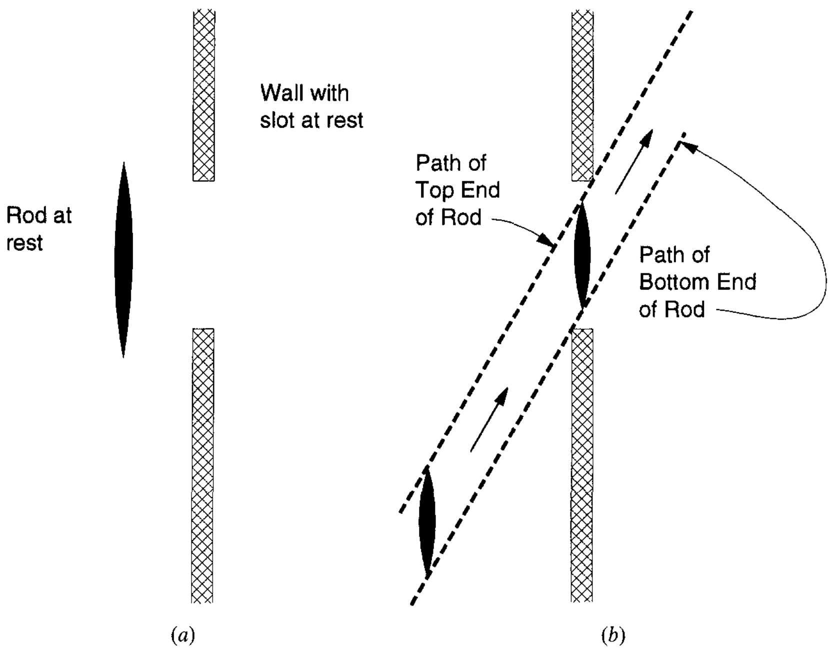


Fig. 2.1. (*a*) When the rod and the wall are both at rest, the rod is longer than the slot. (*b*) When the rod is given a high velocity in a direction parallel to the wall with a small sideways drift, the rod shortens and passes through the slot in the stationary wall. What happens in the rest frame of the rod where the length of the slot shrinks and the length of the rod remains the same?

contraction factor is $\frac{1}{2}$. I have identified the direction of motion with the x^1-axis. In this case the distance from any point on the wall to the x^2-axis in Fig. 2.2b is half the corresponding distance in Fig. 2.2a. Furthermore, the distance of any point on the rod to the x^2-axis in Fig. 2.2b is twice the corresponding distance in Fig. 2.2a. What becomes clear when the two figures are compared is that what is parallel in one frame may not be parallel in another frame and events which are simultaneous in one frame may not be simultaneous in another frame. In the rest frame of the wall, the event of the top end of the rod passing the top end of the slot is simultaneous with the event of the bottom end of the rod passing the bottom end of the slot. These two events are not simultaneous in the rest frame of the rod.

Problem 2.1. Show that $\exp(\hat{\gamma}_{10}\phi) = I\cosh\phi + \hat{\gamma}_{10}\sinh\phi$. Hint: write out the Taylor's series for $\exp(\hat{\gamma}_{10}\phi)$ and then separate the odd and even powers of ϕ into separate sums. Compare the two sums with the Taylor's series for $\cosh\phi$ and $\sinh\phi$.

Problem 2.2. Use the relation that $\exp(\hat{\gamma}_{10}\phi) = I\cosh\phi + \hat{\gamma}_{10}\sinh\phi$ to

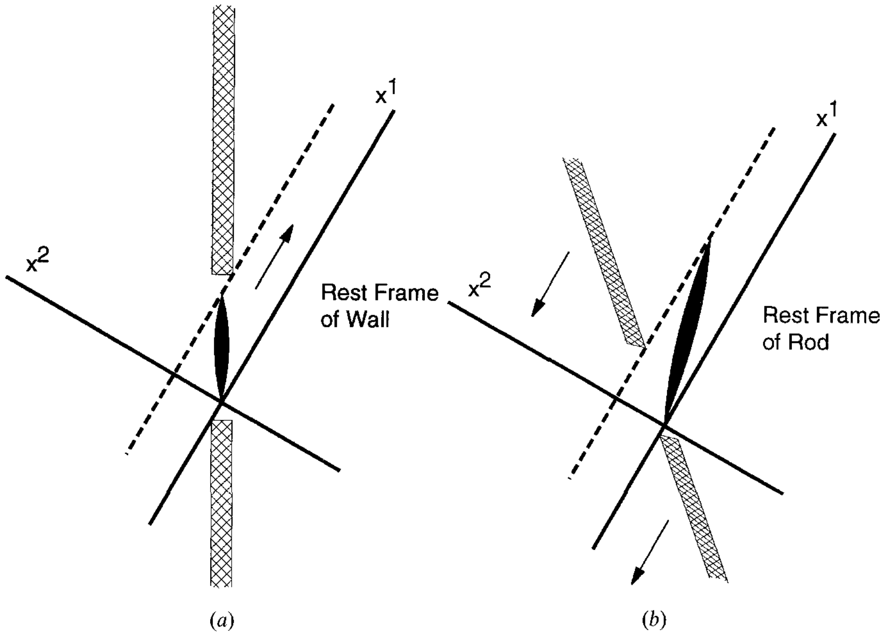


Fig. 2.2. (*a*) The change of lengths occurs in the direction of motion—that is in the direction of the x'-axis. In the rest frame of the wall, the event of the top tip of the rod passing through the plane of the wall is simultaneous with the event of the bottom tip of the rod passing through the same plane. (*b*) Events which are simultaneous in the rest frame of the wall are not simultaneous in the rest frame of the rod.

show that

$$\cosh(\phi_1 + \phi_2) = \cosh \phi_1 \cosh \phi_2 + \sinh \phi_1 \sinh \phi_2$$

and

$$\sinh(\phi_1 + \phi_2) = \sinh \phi_1 \cosh \phi_2 + \cosh \phi_1 \sinh \phi_2.$$

Hint: use the relation that

$$\exp \hat{\gamma}_{10}(\phi_1 + \phi_2) = \exp \hat{\gamma}_{10}\phi_1 \exp \hat{\gamma}_{10}\phi_2.$$

Problem 2.3. Show that the transformation represented by Eqs. (2.14a–d) does indeed preserve the length of a vector as defined in Minkowski 4-space.

Problem 2.4. Suppose $0 < v_1 < c$, $0 < v_2 < c$, and

$$v = \frac{v_1 + v_2}{1 + (v_1 v_2/c^2)}.$$

Show $v < c$. Hint: use the relation $(c - v_1)(c - v_2) > 0$.

Problem 2.5. Show that in general the product of two boosts is not a boost. (To show this, choose two boosts with different directions.) Comment: in general it is possible to show that the product of any number of boosts and rotations can be written as a product of a single boost and a single rotation. A constructive proof of this appears in Appendix A.1.

2.2 Mass, Energy, and Momentum

A basic law of physics is that of the conservation of momentum. This law has a somewhat different flavor in the theory of special relativity than in Newtonian physics.

In Newtonian physics, the directed distance between two points in space can be represented by the 3-vector $(\Delta x, \Delta y, \Delta z)$. Thus the entities Δx, Δy, and Δz transform as components of a 3-vector under a change of coordinates. Since a time interval in Newtonian physics is considered to be a scalar, $\Delta x/\Delta t$, $\Delta y/\Delta t$, and $\Delta z/\Delta t$ can be treated as the components of a legitimate 3-vector $(\Delta x/\Delta t, \Delta y/\Delta t, \Delta z/\Delta t)$ which equals the velocity vector $\boldsymbol{v}$ or (v_x, v_y, v_z).

In Minkowski 4-space, the directed "distance" between two events is represented by the 4-vector $(c\Delta t, \Delta x, \Delta y, \Delta z)$. In special relativity, Δt is no longer a scalar and aside from a factor of c, $\Delta x/\Delta t$, $\Delta y/\Delta t$, and $\Delta z/\Delta t$ are ratios of different components of a 4-vector and are no longer components of a legitimate vector. To obtain something corresponding to a legitimate velocity vector, we must divide $(c\Delta t, \Delta x, \Delta y, \Delta z)$ by a genuine scalar. The most natural scalar for our purposes is the square root of the "length" of the vector. In particular, if

$$\Delta \boldsymbol{s} = \hat{\gamma}_0 c\Delta t + \hat{\gamma}_1 \Delta x + \hat{\gamma}_2 \Delta y + \hat{\gamma}_3 \Delta z,$$

then

$$\begin{aligned}\langle \Delta \boldsymbol{s}, \Delta \boldsymbol{s} \rangle &= c^2(\Delta t)^2 - [(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2] \\ &= c^2(\Delta t)^2[1 - (v/c)^2],\end{aligned}$$

or $\Delta s = c(\Delta t)[1 - (v/c)^2]^{\frac{1}{2}}$. The scalar Δs is known as the *proper time* interval. Thus the special relativistic version of the velocity vector is

$$\begin{aligned}\boldsymbol{u} &= \left(\frac{1}{[1-(v/c)^2]^{\frac{1}{2}}}, \frac{v_x/c}{[1-(v/c)^2]^{\frac{1}{2}}}, \frac{v_y/c}{[1-(v/c)^2]^{\frac{1}{2}}}, \frac{v_z/c}{[1-(v/c)^2]^{\frac{1}{2}}}\right) \\ &= (u^0, u^1, u^2, u^3). \qquad (2.24)\end{aligned}$$

The 4-vector $\boldsymbol{u}$ is known as the *world velocity*. Thus it should not be surprising to the reader to learn that Einstein's version of the momentum vector is the

4-momentum

$$(mu^0, mu^1, mu^2, mu^3) = (mu^0, p^1/c, p^2/c, p^3/c). \tag{2.25}$$

It has been found that under any known physical interaction the sum of the 4-momenta for the interacting particles before the interaction is the same as the sum of the 4-momenta after the interaction. In highly symmetric situations, one can show conservation of the last three components of the 4-momentum is exactly what one would expect from one's knowledge of Newtonian physics and time dilation in special relativity. (See Problem 2.6.)

The faith in this conservation law is so great that if experimentalists do encounter an apparent discrepancy in their data, physicists will hypothesize some previously undetected particle to account for the discrepancy. The neutrino was discovered in just this manner. Wolfgang Pauli (1934) postulated a massless, chargeless particle to account for a 4-momentum discrepancy in beta decay. (There was also a discrepancy in the total spin in the decay process.) Since the postulated particle was both massless and chargeless, it was understood that it would be very difficult to detect. Computations showed that, on the average, a neutrino would penetrate 3,500 light years of solid lead before interacting with matter. Nonetheless, physicists had enough faith in the conservation of 4-momentum to go to extraordinary lengths to detect the neutrino. Eventually, two American physicists Clyde Cowan and Frederick Reines (1953) set up huge detecting tanks near a nuclear-fission reactor at Savannah River, South Carolina. Using some very resourceful techniques, they were able to detect the elusive neutrino. (Technically what they detected was the anti-neutrino. Nonetheless, the conservation of 4-momentum was reaffirmed.)

Let us reexamine the last three components of the 4-momentum vector defined in Eq. (2.25). We immediately see that one can get the special relativistic version of the conservation of momentum by replacing $\boldsymbol{v} = (v_x, v_y, v_z)$ by (u^1, u^2, u^3). However, physicists do not directly measure proper time with their clocks. Therefore it is more natural for most people to measure or compute (v_x, v_y, v_z) than (u^1, u^2, u^3). Thus in the early years of special relativity it made more sense for most physicists to obtain the last three components of the 4-momentum by replacing the mass m in the expression mv by a velocity-dependent *inertial mass* $m_I = m/[1 - (v/c)^2]^{\frac{1}{2}}$. However, in recent years most physicists (at least those in elementary particle theory) have found that their computations are generally cleaner if they carry out their computations in terms of the constant m (the *rest mass*) and the world velocity. This is also more in the spirit of the theory of special relativity.

Till now, I have ignored the first component of the 4-momentum vector. If we expand mu^0 in a power series of $(v/c)^2$, we get $mu^0 = m + \frac{1}{2}m(v/c)^2 +$ higher powers of $(v/c)^2$. Of course $\frac{1}{2}mv^2$ is classically interpreted as the kinetic energy of a particle. Also in Newtonian physics one can always add a

constant to the potential energy of a particle. Thus it is not totally outrageous to interpret $mu^0 = E/c^2$ where E is the energy of the particle. With this identification, we have

$$\left(\frac{m}{(1-(v/c)^2)^{\frac{1}{2}}}, \frac{mv_x/c}{(1-(v/c)^2)^{\frac{1}{2}}}, \frac{mv_y/c}{(1-(v/c)^2)^{\frac{1}{2}}}, \frac{mv_z/c}{(1-(v/c)^2)^{\frac{1}{2}}}\right) = (E/c^2, p^1/c, p^2/c, p^3/c).$$

Computing the "length" of this vector, we have

$$(m)^2 = (E/c^2)^2 - (p/c)^2$$

or

$$E^2 = m^2c^4 + p^2c^2. \tag{2.26}$$

This last equation is considered to be valid even for massless particles such as photons and neutrinos where $m = 0$. In the special case where $v = 0$, one gets the well-known equation $E = mc^2$.

If you wish to deal with complex computations involving multiple boosts and rotations, you will find some useful theorems and computational techniques discussed in Appendices A.1 and A.2.

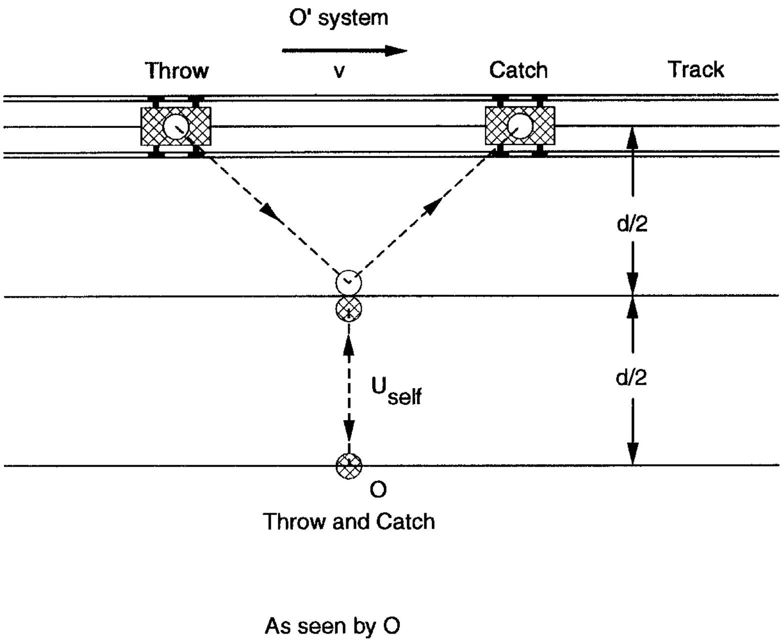
Problem 2.6. (This problem is adapted from the text *Physics of the Atom* by Wehr, Richards, and Adair (1984, pp. 161–165).) Suppose we hand out two idealized basketballs that are perfectly elastic. One ball (ball 2) is given to an observer on a speeding train. The other ball (ball 1) is given to an observer who is placed alongside the track. As the train nears the observer on the ground, each observer is instructed to throw his or her ball with the same velocity in a direction perpendicular to the track in such a way that the basket balls bounce off one another just as the two observers pass one another.

Each observer sees his or her ball behave as if it hit and bounced off a perfectly elastic wall half-way between the two observers. (See Figs. 2.3a, b.)

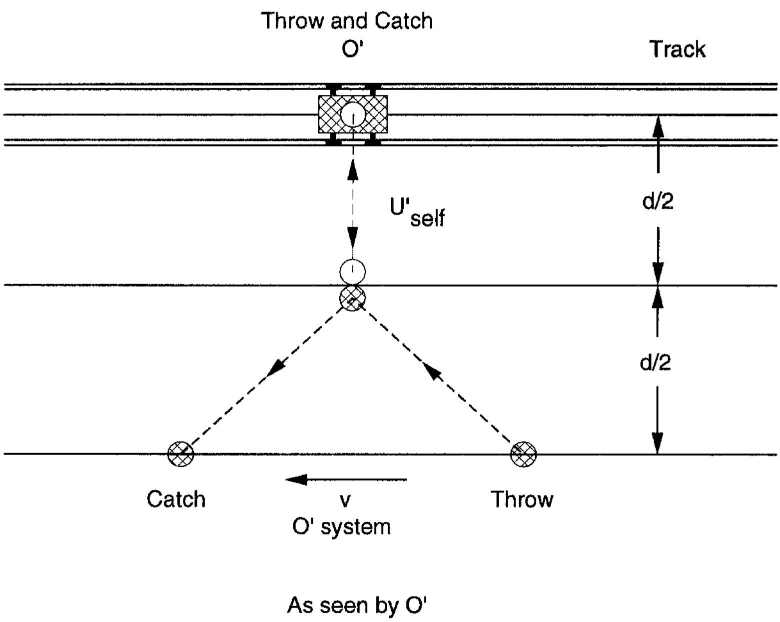
According to the instructions given to the observers, the 4-vectors representing space-time intervals of each ball in the coordinate system of the tosser are:

- ball 1 in the ground (G) system: $(c\Delta t, 0, \Delta y, 0)$ for the outgoing leg and $(c\Delta t, 0, -\Delta y, 0)$ for the return leg;
- ball 2 in the train (T) system: $(c\Delta t, 0, -\Delta y, 0)$ for the outgoing leg and $(c\Delta t, 0, \Delta y, 0)$ for the return leg.

1. Use the boost operator $\mathscr{B}(\phi) = I\cosh(\phi/2) + \hat{\gamma}_{10}\sinh(\phi/2)$ to compute the 4-vectors for the space-time intervals of ball 2 in the G system.



(a)



(b)

Fig. 2.3. Perfectly elastic collision of two identical basketballs as viewed from systems in relative motion. M. Russell Wehr, James A. Richards, Jr., and Thomas W. Adair III, *Physics of the Atom, Fourth Edition* (p. 180), © 1984 by Addison-Wesley Publishing Company, Inc. Reprinted by permission of the publisher.

2. Use the result of part 1 to show that observer G observes the y-component of the velocity for ball 2 to be $(\Delta y/\Delta t)[1-(v/c)^2]^{\frac{1}{2}}$. Suppose observer G has the prejudices of a physicist trained in Newtonian physics. Explain why observer G concludes that ball 2 has a greater mass than ball 1.
3. Show that the sum of the 4-momentum vectors for the two basketballs in the G coordinate system is conserved. (Note: for ball 2 in the G system

$$(v_2)^2 = (v_{\mathrm{T}})^2 + (\Delta y/\Delta t)^2(1-(v_{\mathrm{T}}/c)^2)$$

where v_{T} is the speed of the train. For ball 1 in the G system $(v_1)^2 = (\Delta y/\Delta t)^2$.)

Problem 2.7. Consider a perfectly inelastic collision of two objects of equal mass m which collide head-on with velocities of opposite directions and equal magnitude and then merge into a single mass. Use the conservation of 4-momentum to show that the rest mass of the unified object after the collision is $2m/[1-(v/c)^2]^{\frac{1}{2}}$. (Presumably the excess mass for a macroscopic collision of two clumps of clay would be lost shortly after the collision in the form of heat.)

3

CLIFFORD ALGEBRA FOR FLAT n-DIMENSIONAL SPACES

3.1 Clifford Numbers in n-Dimensional Euclidean or Pseudo-Euclidean Spaces

In Chapter 1, we dealt with Clifford numbers in Euclidean 3-space. In Chapter 2, we dealt with Clifford numbers in Minkowski 4-space. This chapter is devoted to showing how these notions can be extended to "flat" spaces of any finite dimension.

Dirac matrices were introduced by P. A. M. Dirac to formulate his equation for the electron (1928). The matrices introduced by Dirac were specifically designed for Minkowski 4-space. However it is possible to define a generalized version for spaces of arbitrary dimension. This was done by W. K. Clifford over 100 years ago (1878 and 1882).

It is useful to designate a set of n square matrices $\{\hat{\gamma}_1, \hat{\gamma}_2, \hat{\gamma}_3, \ldots, \hat{\gamma}_n\}$ as an *n-dimensional orthonormal Euclidean system of Dirac matrices* if the matrices have the following properties:

1. $\hat{\gamma}_j\hat{\gamma}_k + \hat{\gamma}_k\hat{\gamma}_j = 2\delta_{jk}I$, where I is the identity matrix, $\delta_{jk} = 1$ if $j = k$, and $\delta_{jk} = 0$ if $j \neq k$.
2. By taking all possible products of the n Dirac matrices, one can form a set of 2^n linearly independent matrices. (These products may be written in the form $M_1M_2 \ldots M_n$ where $M_k = \hat{\gamma}_k$ or I.)

It should be noted that one cannot obtain any additional matrices by using the same $\hat{\gamma}_k$ more than once in a product. Since $\hat{\gamma}_j\hat{\gamma}_k = -\hat{\gamma}_k\hat{\gamma}_j$ for $j \neq k$, it is clear that any finite product can be rewritten in the form $\pm(\hat{\gamma}_1)^{k_1}(\hat{\gamma}_2)^{k_2} \ldots (\hat{\gamma}_n)^{k_n}$. Furthermore since $(\hat{\gamma}_m)^{2k} = I$, we can replace $(\hat{\gamma}_m)^{k_m}$ by I if k_m is even or by $\hat{\gamma}_m$ if k_m is odd.

One might ask, "Is it always possible to construct a set of matrices with properties (1) and (2)?" As you might suspect, it is always possible to construct a set of matrices with the desired properties. A method of construction by Kronecker products is discussed in the first section of Chapter 11.

Any linear combination of Dirac matrices is referred to as a *tangent vector* or *1-vector.*

Any linear combination of products of Dirac matrices of the form

$$\sum_{k_1 < k_2 < \quad < k_p} A^{k_1 k_2 \ \cdot k_p} \hat{\gamma}_{k_1} \hat{\gamma}_{k_2} \cdots \hat{\gamma}_{k_p}$$

is referred to as a *p-tangent vector* or more simply as a *p-vector.* A linear combination of p-vectors is a *Clifford number.*

For the first seven chapters of this book, we consider only Clifford numbers for which every coefficient $A^{k_1 k_2 \ \cdot\cdot k_p}$ is real. In this case the Clifford numbers are said to be *real* even though the Dirac matrices may have complex entries. In Chapter 7, which is devoted to a discussion of Dirac's equation for the electron, we also consider Clifford numbers for which the coefficients are complex. In that situation, the Clifford numbers are said to be *complex.*

An *algebra* consists of a vector space V over a field F (usually the set of real numbers $\mathbb{R}$ or the set of complex numbers $\mathbb{C}$) with a binary operation of multiplication such that for all $\alpha \in F$ and $\mathscr{A}$, $\mathscr{B}$, $\mathscr{D} \in V$:

(1) $(\alpha\mathscr{A})\mathscr{B} = \mathscr{A}(\alpha\mathscr{B}) = \alpha(\mathscr{A}\mathscr{B})$;
(2) $(\mathscr{A} + \mathscr{B})\mathscr{D} = \mathscr{A}\mathscr{D} + \mathscr{B}\mathscr{D}$;
(3) $\mathscr{D}(\mathscr{A} + \mathscr{B}) = \mathscr{D}\mathscr{A} + \mathscr{D}\mathscr{B}$.

If, in addition, we have

(4) $(\mathscr{A}\mathscr{B})\mathscr{D} = \mathscr{A}(\mathscr{B}\mathscr{D})$, the algebra is said to be an *associative algebra.*

Clearly the set of Clifford numbers identified with an n-dimensional Euclidean space forms a vector space of dimension 2^n which is closed under matrix multiplication. Thus the set of Clifford numbers forms an associative algebra. If you are genuinely surprised to learn that this algebra is called a *Clifford algebra,* your ability to anticipate the obvious must be suspect.

If the numbers of a Clifford algebra are restricted to real Clifford numbers, the algebra is said to be a *real Clifford algebra.* If the members of a Clifford algebra are allowed to be complex, then that algebra is said to be a *complex Clifford algebra.* (That should not surprise you either.)

We now turn to pseudo-Euclidean spaces. What is a pseudo-Euclidean space? What distinguishes a pseudo-Euclidean space from a Euclidean space is the fact that the scalar product of a non-zero vector with itself is not necessarily positive for a vector in a pseudo-Euclidean space. On the other hand, it is always possible to find an orthonormal basis $\{e_1, e_2, \ldots, e_n\}$ for a pseudo-Euclidean space with the property that

$$\langle e_k, e_k \rangle = \pm 1 \quad \text{for } k = 1, 2, 3, \ldots, n, \tag{3.1}$$

and

$$\langle e_k, e_j \rangle = 0 \quad \text{for } j \neq k. \tag{3.2}$$

An example of a pseudo-Euclidean space is the Minkowski 4-space which was discussed in Chapter 2. One important feature of pseudo-Euclidean spaces is the *signature matrix* n_{jk} where

$$n_{jk} = \langle e_j, e_k \rangle. \tag{3.3}$$

From Eqs. (3.1) and (3.2), it is clear that n_{jk} is a diagonal matrix whose diagonal elements are each ± 1. From a theorem proven by Sylvester (Cartan 1966, pp. 5–6), it is known that regardless of what orthonormal basis is used to span a given pseudo-Euclidean space, the number of positive entries and the number of negative entries on the diagonal of the signature matrix are each invariant integers. For example, regardless of the orthonormal basis chosen to span Minkowski 4-space, one diagonal element of the signature matrix will be $+1$ and the other three diagonal elements will be -1.

A *Clifford algebra for a pseudo-Euclidean space* is generated from a set of Dirac matrices in much the same fashion as it is done for a Euclidean space. We designate a set of n matrices $\{\hat{\gamma}_1, \hat{\gamma}_2, \ldots, \hat{\gamma}_n\}$ as an *orthonormal pseudo-Euclidean system of Dirac matrices* if the members have the following properties:

(1) $\hat{\gamma}_j\hat{\gamma}_k + \hat{\gamma}_k\hat{\gamma}_j = 2n_{jk}I$; (3.4)

(2) by taking all possible products of the n Dirac matrices, one can form a set of 2^n linearly independent matrices.

To construct an n-dimensional system of orthonormal pseudo-Euclidean Dirac matrices, one simply selects an appropriate subset of an n-dimensional orthonormal Euclidean system of Dirac matrices and multiplies each lf them by i, that is by $\sqrt{-1}$.

There are two comments that should be made here. One is that some mathematicians and physicists (mostly English) use the convention that $\hat{\gamma}_j\hat{\gamma}_k + \hat{\gamma}_k\hat{\gamma}_j = -2n_{jk}I$. In particular, this convention is used by Ian R. Porteous (1981, pp. 240–241) and by Roger Penrose and Wolfgang Rindler (1984, p. 124). Also many physicists use a reverse metric for Minkowski 4-space. That is, they use the signature $(-, +, +, +)$ instead of the signature $(+, -, -, -)$.

The second comment is that in his pioneering work on Clifford algebra, Marcel Riesz stated and "proved" a theorem to the effect that property (2) in our definition is implied by property (1) (1958, pp. 10–12). It is now recognized that this theorem is false (Porteous 1981, p. 243). Nonetheless, the method of the proof by Marcel Riesz can be used to prove a corrected version. Restated in a corrected form, the theorem states that if the product $\hat{\gamma}_1\hat{\gamma}_2 \ldots \hat{\gamma}_n$ is not a scalar multiple of the identity matrix I, then property (2) is implied by property (1). This is proven in Section 10.1 of Chapter 11.

When n is even, the product $\hat{\gamma}_1\hat{\gamma}_2 \ldots \hat{\gamma}_n$ anticommutes with $\hat{\gamma}_1$ and thus cannot be a scalar multiple of I. However, for odd values of n and certain signatures, it is possible to have a system of n orthonormal matrices such

that their products span a space of only 2^{n-1} dimensions instead of 2^n dimensions.

In his text *Topological Geometry*, Ian Porteous refers to a system of Clifford numbers generated by a system of n Dirac matrices satisfying property (1) but not necessarily (2) as a Clifford algebra (1981, p. 240). He then refers to a system of Clifford numbers generated by a system of n Dirac matrices satisfying both properties (1) and (2) as a *universal Clifford algebra* (1981, p. 240). Except in Chapter 11, the only Clifford algebras discussed in this book are what Porteous calls universal Clifford algebras.

I am not convinced that so-called nonuniversal Clifford algebras are worthy of serious study. There are two reasons for this. For those interested in the geometric aspect of Clifford algebras, the geometric interpretation of p-vectors becomes nonsense for nonuniversal Clifford algebras. For those interested in the algebraic aspect of Clifford algebras, it should be observed that any nonuniversal Clifford algebra stemming from a Euclidean or pseudo-Euclidean space of n dimensions is isomorphic to some universal Clifford algebra stemming from a Euclidean or pseudo-Euclidean space of $n - 1$ dimensions. This is a point that should not be ignored by the serious physicist.

In treatments of Dirac's equation, the product $\pm\hat{\gamma}_0\hat{\gamma}_1\hat{\gamma}_2\hat{\gamma}_3$ or $\pm i\hat{\gamma}_0\hat{\gamma}_1\hat{\gamma}_2\hat{\gamma}_3$ is frequently labeled as γ_5. In this context, it is understood that γ_5 is a notational shorthand for a product of Dirac matrices. However, some physicists have pushed this labeling beyond such a notational shorthand. In a partially successful attempt to incorporate gravitational and electromagnetic fields into a single metric, Theodor Kaluza (1921), introduced a fifth dimension. With that thought in mind, some physicists have attempted to modify Dirac's equation for the electron by using one of the γ_5's defined above. However, the resulting complex Clifford algebra is identical to the complex Clifford algebra associated with Minkowski 4-space.

I don't know enough about these theories to say with complete confidence that this use of γ_5 is wrong. After all, the fifth dimension in Kaluza's theory is not quite on a par with the other four dimensions. However, the physicist using γ_5 in this manner should be aware of the option of using a universal Clifford algebra associated with a 5-dimensional pseudo-Euclidean vector space.

Problem 3.1. Suppose $\hat{\gamma}_k = \sigma_k$ for $k = 1$, 2, and 3. Explain why this set of Dirac matrices can be used to generate the real universal Clifford algebra for Euclidean 3-space but not the complex universal Clifford algebra for Euclidean 3-space. (For definition of the three Pauli matrices, see Problem 1.2.)

3.2 Dirac Matrices in Real Euclidean or Pseudo-Euclidean Spaces

In an n-dimensional real Euclidean space or pseudo-Euclidean space, one can represent any point in space by a linear combination of orthonormal

Dirac matrices which we will refer to as a *position vector*:

$$s = \hat{\gamma}_\alpha x^\alpha. \tag{3.5}$$

Note: unless specifically indicated otherwise, we will use the standard Einstein convention in which a repeated index appearing in both upper and lower positions indicates a summation. In the expression above:

$$s = \sum_{\alpha=1}^{n} \hat{\gamma}_\alpha x^\alpha.$$

Corresponding to any given coordinate system $\{u^1, u^2, \ldots, u^n\}$ there is a corresponding system of n linearly independent Dirac matrices which will be referred to as a *coordinate system of Dirac matrices*:

$$\gamma_\alpha = \frac{\partial s}{\partial u^\alpha} = \hat{\gamma}_\beta \left(\frac{\partial x^\beta}{\partial u^\alpha}\right) \quad \text{for } \alpha = 1, 2, \ldots, n. \tag{3.6}$$

To indicate a coordinate system, we will use Greek indices and omit the caps on the Dirac matrices. If the system is not necessarily a coordinate system, we will use Latin indices. For those situations where we specifically wish to indicate an orthonormal system of Dirac matrices, we will continue to use caps on the Dirac matrices.

A useful example of a coordinate system in Euclidean 3-space other than the usual $\{x, y, z\}$ system is the spherical coordinate system. For this system, we have

$$s = \hat{\gamma}_1 x + \hat{\gamma}_2 y + \hat{\gamma}_3 z = \hat{\gamma}_1 r \cos\phi \sin\theta + \hat{\gamma}_2 r \sin\phi \sin\theta + \hat{\gamma}_3 r \cos\theta.$$

From Eq. (3.6), it is clear that the coordinate Dirac matrices are

$$\gamma_r = \frac{\partial s}{\partial r} = \hat{\gamma}_1 \cos\phi \sin\theta + \hat{\gamma}_2 \sin\phi \sin\theta + \hat{\gamma}_3 \cos\theta, \tag{3.7}$$

$$\gamma_\theta = \frac{\partial s}{\partial \theta} = \hat{\gamma}_1 r \cos\phi \cos\theta + \hat{\gamma}_2 r \sin\phi \cos\theta - \hat{\gamma}_3 r \sin\theta \tag{3.8}$$

and

$$\gamma_\phi = \frac{\partial s}{\partial \phi} = -\hat{\gamma}_1 r \sin\phi \sin\theta + \hat{\gamma}_2 r \cos\phi \sin\theta. \tag{3.9}$$

For any kind of reasonable coordinate system, the system of matrices generated by Eq. (3.6) can be used as a basis for an n-dimensional vector space at most points. However, at some points this may not be true. For example in the case of the spherical coordinate system $\gamma_\phi = 0$ for any point

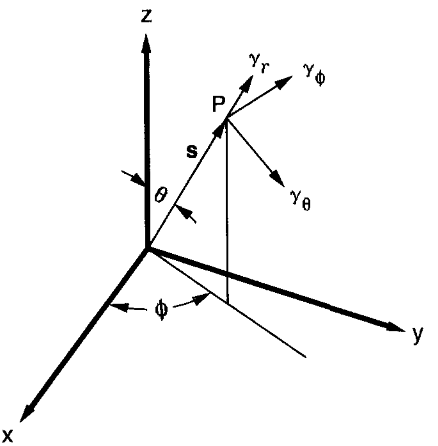


Fig. 3.1. The direction and the length of the spherical coordinate Dirac matrices depend on the particular point with which they are associated.

on the z-axis. Thus the spherical coordinate system of Dirac matrices does not span a 3-dimensional space at those particular points.

Usually both the magnitude and direction of the γ_i's will be dependent on the spatial coordinates and therefore a given set of γ_i's will be identified with a particular point in space. This is clearly true in the spherical coordinate system just discussed. The γ_i's take on meaning only when they are evaluated at a particular point in space.

Physicists frequently encounter vectors which are functions of their location. One obvious example is the vector used to represent an electric field. For this reason it might be natural to refer to the γ_i's as "field vectors." However geometers have adopted an alternate terminology. Following their terminology, the vector space spanned by the γ_i's identified with a given point is called the *tangent space* at that point. A vector in that space is described as a *tangent vector*.

To get a physical insight into these notions, it may be useful to refer to Fig. 3.1. Point P is defined by the position vector s. Furthermore the three tangent vectors γ_r, γ_θ, and γ_ϕ span a 3-dimensional tangent space at the point P.

At first thought these "tangent vectors" may not appear to be tangent to anything. However, at the point (r_0, θ_0, ϕ_0), γ_θ and γ_ϕ are tangent to the sphere $r = r_0$. The Dirac matrices γ_r and γ_ϕ are tangent to the cone $\theta = \theta_0$. And finally γ_r and γ_θ are tangent to the plane $\phi = \phi_0$.

For the spaces discussed so far, the coordinate basis is an orthogonal basis. However this is not true in general. This will be discussed further in Chapter 5.

Problem 3.2. For the 3-dimensional Euclidean space, compute the system of Dirac matrices corresponding to the cylindrical coordinate system $\{\rho, \theta, z\}$

where in terms of the usual Cartesian coordinates $\{x, y, z\}$, we have $x = \rho \cos \theta$, $y = \rho \sin \theta$, and $z = z$.

Problem 3.3. Suppose $\boldsymbol{a} = \gamma_j A^j$ and $\boldsymbol{b} = \gamma_k B^k$. Show that the symmetric form $\boldsymbol{ab} + \boldsymbol{ba}$ is a scalar (a 0-vector). Hint: use Eq. (3.6) along with Eq. (3.4).

3.3 The Metric Tensor and the Scalar Product for 1-Vectors

An arbitrary vector $\boldsymbol{a}$ can be written in the form $\boldsymbol{a} = A^j\gamma_j$. The result of Problem 3.3 makes it possible to define a scalar product $\langle \boldsymbol{a}, \boldsymbol{b} \rangle$ of two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$. We note that since $\boldsymbol{ab} + \boldsymbol{ba}$ is a 0-vector, we can define $\langle \boldsymbol{a}, \boldsymbol{b} \rangle$ as the coefficient of I that occurs in the following equation:

$$\tfrac{1}{2}(\boldsymbol{ab} + \boldsymbol{ba}) = \langle \boldsymbol{a}, \boldsymbol{b} \rangle I = \langle \boldsymbol{b}, \boldsymbol{a} \rangle I. \tag{3.10}$$

Two symmetric matrices arise from this definition of the scalar product of two 1-vectors. One is the signature matrix already encountered in Section 3.1 which is related to an orthonormal basis as follows:

$$n_{jk} = n_{kj} = \langle \hat{\gamma}_j, \hat{\gamma}_k \rangle. \tag{3.11}$$

The second symmetric matrix is known as the *metric tensor* $g_{\alpha\beta}$. This matrix arises from a given coordinate basis of Dirac matrices. In particular:

$$g_{\alpha\beta} = g_{\beta\alpha} = \langle \gamma_\alpha, \gamma_\beta \rangle. \tag{3.12}$$

For the flat Euclidean or pseudo-Euclidean spaces discussed so far, the transformation used to obtain the coordinate system of Dirac matrices from the signature matrix is a simple matter. In particular since

$$\gamma_\alpha = \hat{\gamma}_j \left(\frac{\partial x^j}{\partial u^\alpha} \right),$$

Eq. (3.12) can be rewritten as

$$g_{\alpha\beta} = \langle \hat{\gamma}_j, \hat{\gamma}_k \rangle \left(\frac{\partial x^j}{\partial u^\alpha} \right) \left(\frac{\partial x^k}{\partial u^\beta} \right)$$

or

$$g_{\alpha\beta} = n_{jk} \frac{\partial x^j}{\partial u^\alpha} \frac{\partial x^k}{\partial u^\beta}. \tag{3.13}$$

The inverse of the matrix $g_{\alpha\beta}$ is designated by $g^{\alpha\beta}$. Since $g_{\alpha\beta}$ is symmetric with respect to its two lower indices, the inverse matrix is also symmetric with respect to its two upper indices. That is, $g^{\alpha\beta} = g^{\beta\alpha}$.

This inverse matrix can be used to construct a biorthogonal basis

for the space of tangent vectors at any given point. This alternate basis $\{\gamma^1, \gamma^2, \ldots, \gamma^n\}$ is designated by indices in the upper position and is defined by the relation:

$$\gamma^\alpha = g^{\alpha\beta}\gamma_\beta \quad \text{for } \alpha = 1, 2, \ldots, n. \tag{3.14}$$

We note that $\langle\gamma^\alpha, \gamma_\beta\rangle = g^{\alpha\eta}\langle\gamma_\eta, \gamma_\beta\rangle = g^{\alpha\eta}g_{\eta\beta}$ or

$$\langle\gamma^\alpha, \gamma_\beta\rangle = \delta^\alpha_\beta, \tag{3.15}$$

where

$$\delta^\alpha_\beta = \begin{cases} 1 & \text{if } \alpha = \beta \\ 0 & \text{if } \alpha \neq \beta \end{cases}. \tag{3.16}$$

To distinguish the γ^α's from the γ_α's, we will label the γ^α's as *upper index Dirac matrices* and the γ_α's as *lower index Dirac matrices*.

It is because of Eq. (3.15) that the two bases are said to be *biorthogonal.*

The distinction between the two bases is particularly important for coordinate systems of Dirac matrices. In that case, the position of the indices determines how the Dirac matrices are transformed under a change of coordinates. Consider a change of coordinates from $\{u^1, u^2, \ldots, u^n\}$ to $\{\bar{u}^1, \bar{u}^2, \ldots, \bar{u}^n\}$.

In the two coordinate systems:

$$\mathrm{d}s = \gamma_\alpha\, \mathrm{d}u^\alpha = \bar{\gamma}_\alpha\, \mathrm{d}\bar{u}^\alpha.$$

Thus

$$\gamma_\alpha = \frac{\partial s}{\partial u^\alpha} \qquad \text{and} \qquad \bar{\gamma}_\beta = \frac{\partial s}{\partial \bar{u}^\beta}.$$

Since

$$\frac{\partial s}{\partial \bar{u}^\beta} = \frac{\partial u^\alpha}{\partial \bar{u}^\beta}\frac{\partial s}{\partial u^\alpha},$$

it is clear that

$$\bar{\gamma}_\beta = \frac{\partial u^\alpha}{\partial \bar{u}^\beta}\gamma_\alpha. \tag{3.17}$$

It also immediately follows that

$$\bar{g}_{\alpha\beta} = \langle\bar{\gamma}_\alpha, \bar{\gamma}_\beta\rangle = \langle\gamma_\eta, \gamma_\nu\rangle \frac{\partial u^\eta}{\partial \bar{u}^\alpha}\frac{\partial u^\nu}{\partial \bar{u}^\beta}$$

or

$$\bar{g}_{\alpha\beta} = g_{\eta\nu} \frac{\partial u^\eta}{\partial \bar{u}^\alpha} \frac{\partial u^\nu}{\partial \bar{u}^\beta}. \tag{3.18}$$

By contrast, it will be demonstrated that

$$\bar{g}^{\alpha\beta} = g^{\eta\nu} \frac{\partial \bar{u}^\alpha}{\partial u^\eta} \frac{\partial \bar{u}^\beta}{\partial u^\nu}. \tag{3.19}$$

To verify Eq. (3.19), we need to show that the right-hand side of Eq. (3.19) is the inverse matrix of $\bar{g}_{\beta\theta}$. That is, we must show that

$$\delta^\alpha_\theta = \bar{g}_{\beta\theta} g^{\eta\nu} \frac{\partial \bar{u}^\alpha}{\partial u^\eta} \frac{\partial \bar{u}^\beta}{\partial u^\nu}.$$

But this is equivalent to the equation:

$$\delta^\alpha_\theta = \left(g_{\xi\phi} \frac{\partial u^\xi}{\partial \bar{u}^\beta} \frac{\partial u^\phi}{\partial \bar{u}^\theta} \right) g^{\eta\nu} \frac{\partial \bar{u}^\alpha}{\partial u^\eta} \frac{\partial \bar{u}^\beta}{\partial u^\nu}. \tag{3.20}$$

To condense the right-hand side of Eq. (3.20), we note that

$$\begin{aligned}
\left(g_{\xi\phi} \frac{\partial u^\xi}{\partial \bar{u}^\beta} \frac{\partial u^\phi}{\partial \bar{u}^\theta} \right) g^{\eta\nu} \frac{\partial \bar{u}^\alpha}{\partial u^\eta} \frac{\partial \bar{u}^\beta}{\partial u^\nu} &= g_{\xi\phi} g^{\eta\nu} \frac{\partial u^\phi}{\partial \bar{u}^\theta} \frac{\partial \bar{u}^\alpha}{\partial u^\eta} \left(\frac{\partial u^\xi}{\partial \bar{u}^\beta} \frac{\partial \bar{u}^\beta}{\partial u^\nu} \right) \\
&= g_{\xi\phi} g^{\eta\nu} \frac{\partial u^\phi}{\partial \bar{u}^\theta} \frac{\partial \bar{u}^\alpha}{\partial u^\eta} \left(\frac{\partial u^\xi}{\partial u^\nu} \right) \\
&= g_{\xi\phi} g^{\eta\nu} \frac{\partial u^\phi}{\partial \bar{u}^\theta} \frac{\partial \bar{u}^\alpha}{\partial u^\eta} (\delta^\xi_\nu) \\
&= g_{\nu\phi} g^{\eta\nu} \frac{\partial u^\phi}{\partial \bar{u}^\theta} \frac{\partial \bar{u}^\alpha}{\partial u^\eta} \\
&= \delta^\eta_\phi \frac{\partial u^\phi}{\partial \bar{u}^\theta} \frac{\partial \bar{u}^\alpha}{\partial u^\eta} \\
&= \frac{\partial u^\eta}{\partial \bar{u}^\theta} \frac{\partial \bar{u}^\alpha}{\partial u^\eta} \\
&= \frac{\partial \bar{u}^\alpha}{\partial \bar{u}^\theta} \\
&= \delta^\alpha_\theta.
\end{aligned}$$

Thus we have verified Eq. (3.20) and thereby Eq. (3.19).

Using the fact that $\bar{\gamma}^{\alpha} = \bar{g}^{\alpha\beta}\bar{\gamma}_{\beta}$, it is not too difficult to show that

$$\bar{\gamma}^{\alpha} = \gamma^{\beta} \frac{\partial \bar{u}^{\alpha}}{\partial u^{\beta}}. \tag{3.21}$$

(See Problem 3.4.)

In a similar fashion, the reader should not have difficulty demonstrating that if

$$\boldsymbol{a} = A^{\alpha}\gamma_{\alpha} = A_{\beta}\gamma^{\beta} = \bar{A}^{\nu}\bar{\gamma}_{\nu} = \bar{A}_{\eta}\bar{\gamma}^{\eta},$$

then

$$\bar{A}^{\nu} = A^{\alpha} \frac{\partial \bar{u}^{\nu}}{\partial u^{\alpha}} \tag{3.22}$$

and

$$\bar{A}_{\eta} = A_{\beta} \frac{\partial u^{\beta}}{\partial \bar{u}^{\eta}}. \tag{3.23}$$

It is important to note the position of the indices and the bars in Eqs. (3.17), (3.18), (3.19), and (3.21). We observe that when we change the system of coordinates, the upper index components A^{α} of a vector $\boldsymbol{a}$ undergo the same transformation as the upper index Dirac matrices. Similarly the lower index components A_{α} of a vector $\boldsymbol{a}$ undergo the same transformation as the lower index Dirac matrices.

In the presence of a non-singular metric, the upper index Dirac matrices span the same space as the lower index Dirac matrices. In the absence of a nonsingular metric one can no longer use Eq. (3.14) to define upper index Dirac matrices.

It is interesting to observe that even in the absence of any metric where there is no Clifford algebra some of the mathematical structure can be retained—at least in an analogous form. In the absence of a metric, one must use the formalism of differential forms.

It is worthwhile to compare the formalism of differential forms with that of Clifford algebra. In the formalism of differential forms, one represents a tangent vector as a linear combination of the entities $(\partial/\partial x^{\beta})$. These entities are analogous to a coordinate system of Dirac matrices and undergo the same transformation when a change is made in the coordinate system. In the formalism of Clifford algebra, one could define the upper index Dirac matrices by the requirement that $\langle \gamma^{\alpha}, \gamma_{\beta} \rangle = \delta^{\alpha}_{\beta}$. In the formalism of differential forms, an analogue of the upper index Dirac matrices is introduced in a slightly different way. In place of γ^{α}, one introduces a *1-form* $\mathrm{d}x^{\alpha}$ which is defined as a linear mapping from the space of tangent vectors to the set of real numbers. In particular

$$\mathrm{d}x^{\alpha}\colon \partial/\partial x^{\beta} \to \delta^{\alpha}_{\beta} \qquad \text{so} \qquad \mathrm{d}x^{\alpha}\colon A^{\beta}(\partial/\partial x^{\beta}) \to A^{\alpha}. \tag{3.24}$$

These 1-forms span an alternate vector space with the same dimension as the space of tangent vectors. Rather than being biorthogonal, the 1-forms dx^α form a basis which is said to be *dual* to the coordinate basis formed by the $(\partial/\partial x^\beta)$'s. Furthermore the space spanned by 1-forms is also said to be dual to the space of tangent vectors.

In the formalism of Clifford algebra there are two natural ways of representing the same vector for a given coordinate system:

$$\boldsymbol{a} = A^\alpha \gamma_\alpha = A_\beta \gamma^\beta.$$

From the relation $\gamma^\beta = g^{\alpha\beta}\gamma_\alpha$, the reader can show that $A^\alpha = g^{\alpha\beta}A_\beta$.

When a non-singular metric is introduced into the formalism of differential forms, one has a natural isomorphism between the coordinate basis of the space of tangent vectors and the coordinate basis of the space of 1-forms. Namely:

$$dx^\alpha \leftrightarrow g^{\alpha\beta}(\partial/\partial x^\beta).$$

Under this isomorphism, the tangent vector $A^\alpha(\partial/\partial x^\beta)$ is mapped onto the 1-form $A_\beta(dx^\beta)$ where

$$A^\alpha = g^{\alpha\beta}A_\beta.$$

In both formalisms, a change in coordinates implies a transformation in the components A^α and A_β given by Eqs. (3.22) and (3.23).

To distinguish the way they behave under a change of coordinates, the A_α's are described as being *components of a covariant tensor of order one.* By contrast the A^α's are described as the *components of a contravariant tensor of order one.* In the same spirit, the components of the metric tensor are *components of a covariant tensor of order two.*

In the context of Clifford algebra, this terminology is a little strange because the A_α's and the A^α's may be components of the same entity.

Generally we must be sensitive to the order of the indices for the components of a covariant (or contravariant tensor). Generally $A_{\alpha\beta} \neq A_{\beta\alpha}$. For this reason, we must be sensitive about the vertical alignment of the indices for a *mixed* tensor. We cannot assume that $A_\alpha{}^\beta = A^\beta{}_\alpha$. We note that $g_{\eta\beta}A_\alpha{}^\beta = A_{\alpha\eta}$ while $g_{\eta\beta}A^\beta{}_\alpha = A_{\eta\alpha}$. Thus if $A_{\alpha\eta} \neq A_{\eta\alpha}$ then $A_\alpha{}^\beta \neq A^\beta{}_\alpha$.

One exceptional case where we do not have to make this distinction is the Kronecker delta symbol. Since $\delta_\alpha{}^\beta = \delta^\beta{}_\alpha$, we will save space by simply writing δ_α^β. We will also align the indices in a similar manner when we encounter a generalized version of the Kronecker delta symbol in the next section.

Another exception also appears in the next section where we use a combination of coordinate and noncoordinate indices.

Problem 3.4. Use the fact that $\bar{\gamma}^\alpha = \bar{g}^{\alpha\beta}\bar{\gamma}_\beta$ to verify Eq. (3.21).

Problem 3.5. Show that if $A_\alpha \gamma^\alpha = A^\beta \gamma_\beta$, then

$$A^\beta = g^{\alpha\beta} A_\beta \tag{3.25}$$

and

$$A_\alpha = g_{\alpha\beta} A^\beta. \tag{3.26}$$

Because of Eqs. (3.25) and (3.26), $g^{\alpha\beta}$ and $g_{\alpha\beta}$ are sometimes referred to respectively as the *index raising operator* and the *index lowering operator.*

Problem 3.6. Prove $\langle \gamma^\alpha, \gamma^\beta \rangle = g^{\alpha\beta}$.

Problem 3.7. Components of a tensor can be characterized by their transformation properties. For example $A^\eta{}_\nu$ is a component of a mixed tensor of order two if

$$\bar{A}^\alpha{}_\beta = A^\eta{}_\nu \frac{\partial u^\nu}{\partial \bar{u}^\beta} \frac{\partial \bar{u}^\alpha}{\partial u^\eta}.$$

Using that criteria show δ^α_β is a component of a mixed tensor of order two.

Problem 3.8. Show that if $\boldsymbol{a} = A^\nu \gamma_\nu$ and $\boldsymbol{b} = B^\eta \gamma_\eta$, then $\langle \boldsymbol{a}, \boldsymbol{b} \rangle = A^\nu B^\eta g_{\nu\eta} = A_\alpha B_\beta g^{\alpha\beta} = A^\beta B_\beta = A_\alpha B^\alpha$.

Problem 3.9. Use Eq. (3.13) to show that for spherical coordinates $g_{rr} = 1$, $g_{\theta\theta} = r^2$, $g_{\phi\phi} = r^2 \sin^2\theta$, and $g_{jk} = 0$ if $j \neq k$.

Problem 3.10. Compute the inverse of the metric tensor computed in Problem 3.9. Then use the result and Eq. (3.14) to compute γ^r, γ^θ, and γ^ϕ in terms of $\hat{\gamma}_1$, $\hat{\gamma}_2$, and $\hat{\gamma}_3$.

Problem 3.11. Show that for Euclidean coordinates

$$\left(\gamma^1 \frac{\partial}{\partial x^1} + \gamma^2 \frac{\partial}{\partial x^2} + \gamma^3 \frac{\partial}{\partial x^3}\right)^2 = I\left[\frac{\partial^2}{(\partial x^1)^2} + \frac{\partial^2}{(\partial x^2)^2} + \frac{\partial^2}{(\partial x^3)^2}\right].$$

Problem 3.12. Show that if

$$\bar{\gamma}^\alpha = \gamma^\beta \frac{\partial \bar{u}^\alpha}{\partial u^\beta},$$

then

$$\gamma^\beta \frac{\partial}{\partial u^\beta} = \bar{\gamma}^\alpha \frac{\partial}{\partial \bar{u}^\alpha}.$$

Problem 3.13. Show that

$$\left(\gamma^r \frac{\partial}{\partial r} + \gamma^\theta \frac{\partial}{\partial \theta} + \gamma^\phi \frac{\partial}{\partial \phi}\right)^2$$
$$= I\left[\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2}{\partial \phi^2}\right].$$

(Use the results from Problem 3.19.)

Problem 3.14. For cylindrical coordinates $x^1 = \rho\cos\theta$, $x^2 = \rho\sin\theta$, and $x^3 = z$, compute γ_ρ, γ_θ, and γ_z in terms of $\hat{\gamma}_1$, $\hat{\gamma}_2$, and $\hat{\gamma}_3$. Use this result to compute the components of the metric tensor and then the components of the inverse tensor. Then compute γ^ρ, γ^θ, and γ^z. Finally show that

$$\left[\gamma^\rho \frac{\partial}{\partial \rho} + \gamma^\theta \frac{\partial}{\partial \theta} + \gamma^z \frac{\partial}{\partial z}\right]^2 = I\left[\frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}\right].$$

Problem 3.15. An *index free Clifford number* $\mathscr{A}$ is a Clifford number which contains no unsummed indices. An example of such a Clifford number is the 3-vector $\mathscr{A} = A^{\alpha\beta}{}_\nu \gamma_\alpha \gamma_\beta \gamma^\nu$, where $A^{\alpha\beta}{}_\nu$ is a component of a tensor. Demonstrate that an index free Clifford number transforms under a change of coordinates as a scalar, that is, $\bar{\mathscr{A}} = \bar{A}^{\phi\eta}{}_\xi \bar{\gamma}_\phi \bar{\gamma}_\eta \bar{\gamma}^\xi = \mathscr{A} = A^{\alpha\beta}{}_\nu \gamma_\alpha \gamma_\beta \gamma^\nu$.

3.4 The Exterior Product for p-Vectors and the Scalar Product for Clifford Numbers

As already indicated in Section 3.1, a product of p distinct orthonormal Dirac matrices or a linear combination of such products is referred to as a p-vector. However, this does not mean that the product of p distinct 1-vectors is necessarily a p-vector. For example, suppose $\hat{\gamma}_1$, $\hat{\gamma}_2$, and $\hat{\gamma}_3$ are orthonormal with $(\hat{\gamma}_1)^2 = (\hat{\gamma}_2)^2 = (\hat{\gamma}_3)^2 = I$. Then if

$$\boldsymbol{a} = A^1\hat{\gamma}_1 + A^2\hat{\gamma}_2 + A^3\hat{\gamma}_3 \qquad \text{and} \qquad \boldsymbol{b} = B^1\hat{\gamma}_1 + B^2\hat{\gamma}_2 + B^3\hat{\gamma}_3,$$

we have

$$\boldsymbol{ab} = (A^1B^1 + A^2B^2 + A^3B^3)I + (A^2B^3 - A^3B^2)\hat{\gamma}_2\hat{\gamma}_3$$
$$+ (A^3B^1 - A^1B^3)\hat{\gamma}_3\hat{\gamma}_1 + (A^1B^2 - A^2B^1)\hat{\gamma}_1\hat{\gamma}_2.$$

Thus it is not difficult to see that in general the product of two 1-vectors is a linear combination of a 2-vector and a 0-vector or scalar. However, a 2-vector can be constructed by taking a linear combination of $\boldsymbol{ab}$ and $\boldsymbol{ba}$. In

particular

$$\tfrac{1}{2}(\boldsymbol{ab} - \boldsymbol{ba}) = (A^2B^3 - A^3B^2)\hat{\gamma}_2\hat{\gamma}_3 + (A^3B^1 - A^1B^3)\hat{\gamma}_3\hat{\gamma}_1 + (A^1B^2 - A^2B^1)\hat{\gamma}_1\hat{\gamma}_2.$$

More generally, suppose we consider p distinct Dirac matrices each one of which is expanded in terms of an n-dimensional orthonormal basis; that is, $\gamma_\nu = W^j_\nu\hat{\gamma}_j$ for $\nu = 1, 2, \ldots, p$. Then

$$\gamma_1\gamma_2 \ldots \gamma_p = W^{j_1}_1 W^{j_2}_2 \ldots W^{j_p}_p \hat{\gamma}_{j_1}\hat{\gamma}_{j_2} \ldots \hat{\gamma}_{j_p}. \tag{3.27}$$

The terms on the right side of Eq. (3.27) are of several types. If $\hat{\gamma}_{j_1}, \hat{\gamma}_{j_2}, \ldots, \hat{\gamma}_{j_p}$ are all distinct then their product is a p-vector. On the other hand, if some of the $\hat{\gamma}_{j_k}$'s in a given term are identical then like pairs of the orthonormal Dirac matrices may be grouped together and multiplied out until the only remaining Dirac matrices in that term are all distinct. Thus we see that the right-hand side of Eq. (3.27) consists of a linear combination of p-vectors, $(p-2)$-vectors, $(p-4)$-vectors, and so forth, on down to 1-vectors or 0-vectors.

One can project out the p-vector component on the right-hand side of Eq. (3.27) by antisymmetrizing the product. To do this explicitly, it is useful to introduce the *generalized Kronecker delta symbol* $\delta^{i_1 i_2 \ldots i_p}_{j_1 j_2 \ldots j_p}$. This symbol is defined to be $+1$ if the upper indices are all distinct and the sequence of lower indices is an even permutation of the sequence of the upper indices. The same symbol is equal to -1 if the upper indices are distinct and the sequence of lower indices is an odd permutation of the upper indices. For all other cases, the generalized Kronecker delta symbol is zero.

With this generalized delta function, we can define $\gamma_{\nu_1\nu_2\ldots\nu_p}$ as the antisymmetric product of $\gamma_{\nu_1}, \gamma_{\nu_2}, \ldots, \gamma_{\nu_p}$. In particular:

$$\begin{aligned}
\gamma_{\nu_1\nu_2\ldots\nu_p} &= \left(\frac{1}{p!}\right)\delta^{\eta_1\eta_2\ldots\eta_p}_{\nu_1\nu_2\ldots\nu_p}\gamma_{\eta_1}\gamma_{\eta_2}\ldots\gamma_{\eta_p} \\
&= \left(\frac{1}{p!}\right)\delta^{\eta_1\eta_2\ldots\eta_p}_{\nu_1\nu_2\ldots\nu_p} W^{j_1}_{\eta_1} W^{j_2}_{\eta_2}\ldots W^{j_p}_{\eta_p}\hat{\gamma}_{j_1}\hat{\gamma}_{j_2}\ldots\hat{\gamma}_{j_p} \\
&= \left(\frac{1}{p!}\right)\det\begin{bmatrix} W^{j_1}_{\nu_1} & W^{j_1}_{\nu_2} & \ldots & W^{j_1}_{\nu_p} \\ W^{j_2}_{\nu_1} & W^{j_2}_{\nu_2} & \ldots & W^{j_2}_{\nu_p} \\ \vdots & \vdots & & \\ W^{j_p}_{\nu_1} & W^{j_p}_{\nu_2} & \ldots & W^{j_p}_{\nu_p} \end{bmatrix}\hat{\gamma}_{j_1}\hat{\gamma}_{j_2}\ldots\hat{\gamma}_{j_p}.
\end{aligned} \tag{3.28}$$

It should be noted that if $\gamma_{\nu_1}, \gamma_{\nu_2}, \ldots, \gamma_{\nu_p}$ are linearly dependent, then each determinant on the right-hand side of Eq. (3.28) will be zero and thus the antisymmetric $\gamma_{\nu_1\nu_2\ldots\nu_p}$ will be zero.

For the special case for which $p = n$, the same determinant appears on the right-hand side of Eq. (3.28) $n!$ times. Thus we have

$$\gamma_{12\ldots n} = \det \begin{bmatrix} W_1^1 & W_2^1 & \ldots & W_n^1 \\ W_1^2 & W_2^2 & \ldots & W_n^2 \\ \vdots & \vdots & & \vdots \\ W_1^n & W_2^n & \ldots & W_n^n \end{bmatrix} \hat{\gamma}_1\hat{\gamma}_2 \ldots \hat{\gamma}_n. \tag{3.29}$$

The determinant on the right-hand side of Eq. (3.29) may be considered to be the volume of the n-dimensional parallelepiped spanned by the tangent vectors $\gamma_1, \gamma_2, \ldots, \gamma_n$.

The ordinary matrix product of a p-vector and a q-vector results in a linear combination of vectors of the orders $p + q$, $p + q - 2$, $p + q - 4, \ldots,$ on down to $p - q$. However, for some purposes, it is useful to drop the lower order forms. With this in mind, one defines the *exterior product* of a p-vector and a q-vector as the projection of the ordinary matrix product onto the space of vectors of order $p + q$. In particular

$$\begin{aligned} \gamma_{\nu_1\nu_2\,.\,\nu_p} \wedge \gamma_{\eta_1\eta_2\,..\,\eta_q} &= \frac{1}{(p+q)!}\,\delta^{\beta_1\beta_2\,.\,.\beta_{p+q}}_{\nu_1\nu_2\ldots\nu_p\eta_1\eta_2\,..\,\eta_q}\gamma_{\beta_1\beta_2\ldots\beta_p\beta_{p+1}\beta_{p+2}\,..\,\beta_{p+q}} \\ &= \gamma_{\nu_1\nu_2\,.\,.\nu_p\eta_1\eta_2\,..\,\eta_q} \end{aligned} \tag{3.30}$$

where it is understood that $\gamma_{\nu_1\nu_2\,..\,\nu_s} = 0$ if any two indices are identical.

This can be generalized a little bit. Suppose

$$\mathscr{F} = \frac{1}{p!}F^{i_1i_2\ldots i_p}\gamma_{i_1i_2\,..\,i_p}$$

and

$$\mathscr{G} = \frac{1}{q!}G^{j_1j_2\,.\,.j_p}\gamma_{j_1j_2\,.\,.j_q}.$$

Then

$$\mathscr{F} \wedge \mathscr{G} = \frac{1}{p!\,q!}F^{i_1i_2\quad i_p}G^{j_1j_2\ldots j_q}\gamma_{i_1i_2\,.\,.i_pj_1j_2\ldots j_q}. \tag{3.31}$$

It should be noted that Eqs. (3.28), (3.29), and (3.30) are valid for both coordinate and noncoordinate systems of Dirac matrices.

Any matrix product or linear combination of such products can be decomposed in a unique way into a linear combination of p-vectors of varying orders. Suppose $\{\hat{\gamma}_1, \hat{\gamma}_2, \ldots, \hat{\gamma}_n\}$ is a set of orthonormal Dirac matrices. Then $\hat{\gamma}_i\hat{\gamma}_j = \hat{\gamma}_{ij}$ if $i \neq j$ and $\hat{\gamma}_i\hat{\gamma}_j = n_{ij}I$ if $i = j$. This can be

summarized by the equation:

$$\hat{\gamma}_i\hat{\gamma}_j = \hat{\gamma}_{ij} + n_{ij}\mathrm{I}. \tag{3.32}$$

(Note: one and only one of the terms on the right-hand side of Eq. (3.32) is nonzero.)

Consider the product $\hat{\gamma}_i\hat{\gamma}_j\hat{\gamma}_k$. If all three indices are distinct then $\hat{\gamma}_i\hat{\gamma}_j\hat{\gamma}_k = \hat{\gamma}_{ijk}$. If the first two indices are identical then $\hat{\gamma}_i\hat{\gamma}_j\hat{\gamma}_k = n_{ij}\hat{\gamma}_k$. If $i \neq j$ and $i = k$, then $\hat{\gamma}_i\hat{\gamma}_j\hat{\gamma}_k = -\hat{\gamma}_j\hat{\gamma}_i\hat{\gamma}_k = -n_{ik}\hat{\gamma}_j$. If $i \neq j$, $i \neq k$, and $j = k$, then $\hat{\gamma}_i\hat{\gamma}_j\hat{\gamma}_k = n_{jk}\hat{\gamma}_i$. If we write

$$\hat{\gamma}_i\hat{\gamma}_j\hat{\gamma}_k = \hat{\gamma}_{ijk} + n_{ij}\hat{\gamma}_k - n_{ik}\hat{\gamma}_j + n_{jk}\hat{\gamma}_i, \tag{3.33}$$

we note that this equation is also valid for the case in which $i = j = k$.

Using similar arguments:

$$\begin{aligned}\hat{\gamma}_i\hat{\gamma}_j\hat{\gamma}_k\hat{\gamma}_m = \hat{\gamma}_{ijkm} &+ n_{ij}\hat{\gamma}_{km} - n_{ik}\hat{\gamma}_{jm} + n_{im}\hat{\gamma}_{jk} \\ &+ n_{jk}\hat{\gamma}_{im} - n_{jm}\hat{\gamma}_{ik} + n_{km}\hat{\gamma}_{ij} + n_{ij}n_{km}I \\ &- n_{ik}n_{jm}I + n_{im}n_{jk}I.\end{aligned} \tag{3.34}$$

To adjust these formulas for a coordinate basis it can be shown that one merely replaces the components of the signature matrix by the components of the metric tensor. To demonstrate this, we observe that

$$\begin{aligned}\gamma_\alpha\gamma_\beta\gamma_\gamma &= W^i_\alpha W^j_\beta W^k_\gamma\hat{\gamma}_i\hat{\gamma}_j\hat{\gamma}_k \\ &= W^i_\alpha W^j_\beta W^k_\gamma[\hat{\gamma}_{ijk} + n_{ij}\hat{\gamma}_k - n_{ik}\hat{\gamma}_j + n_{jk}\hat{\gamma}_i]\end{aligned}$$

or

$$\gamma_\alpha\gamma_\beta\gamma_\nu = \gamma_{\alpha\beta\gamma} + g_{\alpha\beta}\gamma_\nu - g_{\alpha\nu}\gamma_\beta + g_{\beta\nu}\gamma_\alpha. \tag{3.35}$$

Of course any one of the equations written down with lower indices has a corresponding version with upper indices. Corresponding to Eq. (3.28), we have

$$\gamma^{j_1j_2\cdots j_p} = \left(\frac{1}{p!}\right)\delta^{j_1j_2\cdots j_p}_{i_1i_2\cdots i_p}\gamma^{i_1}\gamma^{i_2}\cdots\gamma^{i_p}$$

and corresponding to Eq. (3.35), we have

$$\gamma^\alpha\gamma^\beta\gamma^\nu = \gamma^{\alpha\beta\nu} + g^{\alpha\beta}\gamma^\nu - g^{\alpha\nu}\gamma^\beta + g^{\beta\nu}\gamma^\alpha. \tag{3.36}$$

Since a product of Dirac matrices can be decomposed into a sum of

p-vectors, the same can be said for any Clifford number. For example, consider the spherical coordinate system applied to Euclidean 3-space. Suppose we designate the Dirac matrices by γ_r, γ_θ, and γ_ϕ. Then an arbitrary member $\mathscr{A}$ of the associated Clifford algebra can be written as a linear combination of p-vectors with lower indices up to order 3:

$$\begin{aligned}\mathscr{A} &= AI + A^r\gamma_r + A^\theta\gamma_\theta + A^\phi\gamma_\phi + A^{\theta\phi}\gamma_{\theta\phi}\\ &\quad + A^{\phi r}\gamma_{\phi r} + A^{r\theta}\gamma_{r\theta} + A^{r\theta\phi}\gamma_{r\theta\phi}.\end{aligned} \tag{3.37}$$

Similarly one could write the same Clifford number in terms of p-vectors with upper indices:

$$\begin{aligned}\mathscr{A} &= AI = A_r\gamma^r + A_\theta\gamma^\theta + A_\phi\gamma^\phi + A_{\theta\phi}\gamma^{\theta\phi}\\ &\quad + A_{\phi r}\gamma^{\phi r} + A_{r\theta}\gamma^{r\theta} + A_{r\theta\phi}\gamma^{r\theta\phi}.\end{aligned} \tag{3.38}$$

The set of pure p-vectors can be treated as a vector space. The natural basis for such a vector space is formed by constructing antisymmetric products of p distinct coordinate Dirac matrices. The dimension of this space is $\binom{n}{p}$ since that is the number of ways that p objects can be chosen from a set of size n. For the example above, the space of 2-vectors has dimension $\binom{3}{2} = 3$ and is spanned by $\gamma_{\theta\phi}$, $\gamma_{\phi r}$, and $\gamma_{r\theta}$. The same space is also spanned by the 2-forms $\gamma^{\theta\phi}$, $\gamma^{\phi r}$, and $\gamma^{r\theta}$.

By taking the direct sum of the different p-vector spaces, we arrive at the vector space of all Clifford numbers. As previously indicated the dimension of this larger space must be

$$\sum_p \binom{n}{p} = \sum_p \binom{n}{p} 1^p 1^{n-p} = (1+1)^n = 2^n.$$

In Section 3.3, we defined the scalar product for a pair of 1-vectors. We now turn to the problem of defining the scalar product of any two Clifford numbers. To deal with this, we use two concepts. The first notion is the scalar component of a Clifford number. Following the notation used by David Hestenes and Garret Sobczyk (1984, p. 3), we designate the scalar component of a Clifford number $\mathscr{A}$ by $(\mathscr{A})_0$. For example, from Eq. (3.37) or (3.38), we have $(\mathscr{A})_0 = A$.

The second notion is that of the reverse of a Clifford number. Continuing to follow the notation used by Hestenes and Sobczyk (p. 5) and the terminology of Marcel Riesz (1958, p. 13), we define the *reverse* $\mathscr{A}^\dagger$ of a Clifford number $\mathscr{A}$ to be that Clifford number obtained by reversing the order of all products of Dirac matrices in the linear expansion of $\mathscr{A}$. For example if $\mathscr{A}$ is the Clifford number that appears in Eq. (3.37),

then

$$\begin{aligned}\mathscr{A}^\dagger &= AI + A^r\gamma_r + A^\theta\gamma_\theta + A^\phi\gamma_\phi + A^{\theta\phi}\gamma_{\phi\theta} \\ &\quad + A^{\phi r}\gamma_{r\phi} + A^{r\theta}\gamma_{\theta r} + A^{r\theta\phi}\gamma_{\phi\theta r} \\ &= AI + A^r\gamma_r + A^\theta\gamma_\theta + A^\phi\gamma_\phi - A^{\theta\phi}\gamma_{\theta\phi} \\ &\quad - A^{\phi r}\gamma_{\phi r} - A^{r\theta}\gamma_{r\theta} - A^{r\theta\phi}\gamma_{r\theta\phi}.\end{aligned} \tag{3.39}$$

We can now define the scalar product of two real Clifford numbers $\mathscr{A}$ and $\mathscr{B}$ to be

$$\langle \mathscr{A}, \mathscr{B} \rangle = (\mathscr{A}^\dagger \mathscr{B})_0. \tag{3.40}$$

Problem 3.16. Harley Flanders (1963, p. 14) defines the scalar product of two p-forms in a somewhat different manner. The Clifford algebra analogue would be

$$\langle \gamma^{i_1 i_2 \cdots i_p}, \gamma^{j_1 j_2 \cdots j_p} \rangle = \det \begin{bmatrix} \langle \gamma^{i_1}, \gamma^{j_1} \rangle & \langle \gamma^{i_1}, \gamma^{j_2} \rangle & \cdots & \langle \gamma^{i_1}, \gamma^{j_p} \rangle \\ \langle \gamma^{i_2}, \gamma^{j_1} \rangle & \langle \gamma^{i_2}, \gamma^{j_2} \rangle & \cdots & \langle \gamma^{i_2}, \gamma^{j_p} \rangle \\ \vdots & \vdots & & \vdots \\ \langle \gamma^{i_p}, \gamma^{j_1} \rangle & \langle \gamma^{i_p}, \gamma^{j_2} \rangle & \cdots & \langle \gamma^{i_p}, \gamma^{j_p} \rangle \end{bmatrix}. \tag{3.41}$$

For a slightly more general case, suppose

$$\mathscr{A} = A_{i_1 i_2 \cdots i_p} \gamma^{i_1 i_2 \cdots i_p}$$

and

$$\mathscr{B} = B_{j_1 j_2 \cdots j_p} \gamma^{j_1 j_2 \cdots j_p}$$

then

$$\langle \mathscr{A}, \mathscr{B} \rangle = A_{i_1 i_2 \cdots i_p} B_{j_1 j_2 \cdots j_p} \langle \gamma^{i_1 i_2 \cdots i_p}, \gamma^{j_1 j_2 \cdots j_p} \rangle.$$

Show that for pure p-vectors, the definition of Hestenes and Flanders agree. Suggestion: try using an orthonormal basis.

Problem 3.17. Demonstrate that $(\mathscr{A}^\dagger \mathscr{B})_0 = (\mathscr{B}^\dagger \mathscr{A})_0$ where $\mathscr{A}$ and $\mathscr{B}$ are arbitrary Clifford numbers that are not necessarily index free. (This shows that $\langle \mathscr{A}, \mathscr{B} \rangle = \langle \mathscr{B}, \mathscr{A} \rangle$.)

Problem 3.18. Show that $(\mathscr{A}\mathscr{B})_0 = (\mathscr{B}\mathscr{A})_0$ where $\mathscr{A}$ and $\mathscr{B}$ are not necessarily index free Clifford numbers.

Problem 3.19. Generalize the result of Problem 3.18 to show that the scalar component of a product of several Clifford numbers is invariant under a cyclic permutation, that is

$$\begin{aligned}(\mathscr{A}_1\mathscr{A}_2 \ldots \mathscr{A}_n)_0 &= (\mathscr{A}_2\mathscr{A}_3 \ldots \mathscr{A}_n\mathscr{A}_1)_0 \\ &= (\mathscr{A}_{k+1}\mathscr{A}_{k+2} \ldots \mathscr{A}_n\mathscr{A}_1\mathscr{A}_2 \ldots \mathscr{A}_k). \qquad (3.42)\end{aligned}$$

4

CURVED SPACES EMBEDDED IN HIGHER DIMENSIONAL FLAT SPACES

4.0 Why you may wish to skip Chapter 4

If you have a cursory knowledge of differential geometry, you may wish to skip this chapter. This chapter is devoted to looking at some aspects of m-dimensional surfaces embedded in n-dimensional pseudo-Euclidean spaces. Much of the chapter is further specialized to 2-dimensional surfaces embedded in E^3. We discuss these surfaces to introduce such notions as curvature, Christoffel symbols, and parallel transport. Also on these surfaces, it is possible to motivate the requirement that a derivative operator be torsion free. Nonetheless, if you have encountered these concepts elsewhere, you should be able to skip this chapter without loss of continuity.

4.1 Gaussian Curvature and Parallel Transport on Two-Dimensional Surfaces in E^3

The simplest example of a curved surface is the ordinary 2-dimensional sphere in Euclidean 3-space. Furthermore the geometry of the sphere serves as a major motivation for much of the mathematical work that has been done for more general surfaces. For this reason it is useful to examine some basic results associated with the geometry of a sphere.

Given two points on a surface, a *geodesic* is a path of minimum length passing through the two points.* For a sphere, a geodesic is a segment of a great circle—that is, a circle formed by the intersection of the sphere with

* Most mathematicians will cringe at this definition with good reason. When one travels by plane from New York to London by a great circle route over the Atlantic Ocean, one is following a geodesic. However if one continues the flight over the same great circle, one is still following a geodesic even though the shortest route from London to New York is not over Turkey. Furthermore in Minkowski space, a geodesic is a path of maximum length rather than minimum length. Nevertheless we will rely on the intuition and common sense of the reader until we have developed enough mathematical machinery to give a more sophisticated definition in Section 4.3 of this chapter.

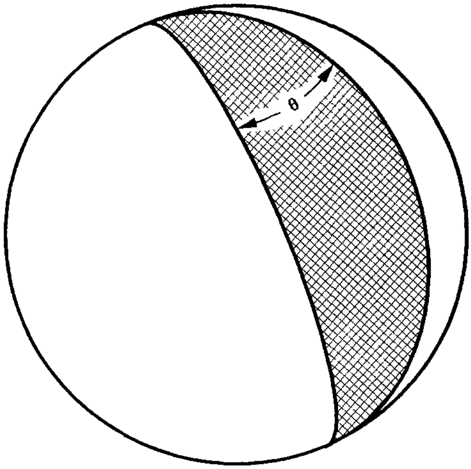


Fig. 4.1. The intersection of two great circles forms a lune.

a plane passing through the center. The intersection of two great circles forms a *lune*. (See Fig. 4.1.)

The area of a sphere is $4\pi r^2$, where r is the radius. The area of a lune is the fraction $\theta/2\pi$ of the sphere. Thus the area of a lune is $(\theta/2\pi)(4\pi r^2)$ or $2r^2\theta$. From this result, it is not too difficult to determine the area of a triangle formed by three great circles. Such a figure is known as a *spherical triangle.*

Theorem. (Refer to Fig. 4.2.) The area of $\triangle$ABC $= [(\text{A} + \text{B} + \text{C}) - \pi]r^2$, where A, B, and C are the vertex angles measured in radians.

Proof. (For shorthand purposes, we will designate the area of a figure simply

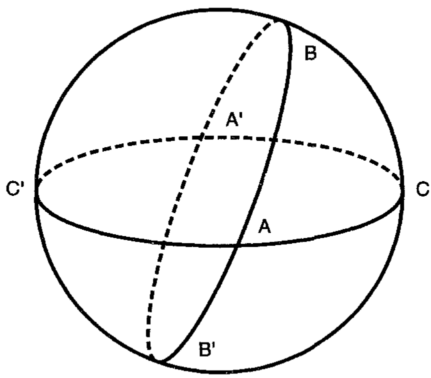


Fig. 4.2. The spherical triangle ABC is formed by three great circles.

by the label for the figure.)

$$\triangle \mathrm{ABC} + \triangle \mathrm{A'BC} = 2r^2\mathrm{A} \quad \text{(this is the area of lune ACA'BA);}$$

$$\triangle \mathrm{ABC} + \triangle \mathrm{ABC'} = 2r^2\mathrm{C};$$

$$\triangle \mathrm{ABC} + \triangle \mathrm{AB'C} = 2r^2\mathrm{B};$$

$$\triangle \mathrm{A'BC'} - \triangle \mathrm{AB'C} = 0 \quad \text{(by symmetry).}$$

We now observe that $\triangle \mathrm{ABC} + \triangle \mathrm{A'BC} + \triangle \mathrm{ABC'} + \triangle \mathrm{A'BC'}$ equals the area of the upper hemisphere in Fig. 4.2 which is $2\pi r^2$. Making use of this fact, we can add up the two sides of the four equations above and obtain the result that

$$2\triangle \mathrm{ABC} + 2\pi r^2 = 2r^2(\mathrm{A} + \mathrm{B} + \mathrm{C}).$$

From this, it immediately follows that

$$\text{Area of } \triangle \mathrm{ABC} = [(\mathrm{A} + \mathrm{B} + \mathrm{C}) - \pi]r^2. \tag{4.1}$$

■

The quantity $1/r^2$ is referred to as the *Gaussian curvature* of the surface and the quantity $(\mathrm{A} + \mathrm{B} + \mathrm{C}) - \pi$ is called the *spherical excess* of $\triangle \mathrm{ABC}$. Thus the product of the area of $\triangle \mathrm{ABC}$ and the Gaussian curvature is the spherical excess. For figures more general than spherical triangles, it is useful to express the product of the area and the Gaussian curvature in slightly different terms.

Suppose we consider the problem of transporting a vector tangent to the spherical surface around the perimeter of the spherical triangle without rotation. When we say "without rotation," we mean without rotation detectable to an observer who is required to make all measurements in the surface of the sphere.

In a flat 2-dimensional plane we can move a vector from point A to point B by sliding it along a straight line while maintaining a constant angle between the vector and the straight line. Such a displacement is called *parallel transport*. (See Fig. 4.3.)

On the surface of a sphere, the analogue of a straight line is a geodesic or great circle. With that in mind, we now try to move a vector around the perimeter of a spherical triangle by parallel transport. (Refer to Fig. 4.4.) Suppose we consider vector $\boldsymbol{v}_1$ tangent to geodesic AB at vertex A. Moving the vector by parallel transport along AB, it becomes vector $\boldsymbol{v}_2$ at vertex B. Continuing the process, the vector becomes $\boldsymbol{v}_3$ at vertex C and vector $\boldsymbol{v}_4$ at the original location vertex A. This is perhaps most easily visualized for the example in which A is the North Pole and BC is a segment of the equator. In this case our parallel transported vector will be pointing South at all times during the motion.

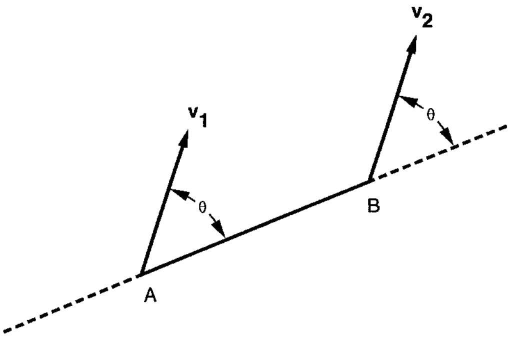


Fig. 4.3. Vector v_2 is the result of a parallel transport of v_1 along the path AB.

Although the vector is moved by parallel transport along all points of the perimeter, we discover that when the vector is returned to its original location, it has become rotated counterclockwise from its original orientation. Referring again to Fig. 4.4, it is not too difficult to convince oneself that this angle of rotation is $(A + B + C) - \pi$ which is exactly the spherical excess of the spherical triangle. Thus if α represents the angle rotated by the vector parallel transported about the complete perimeter of the spherical

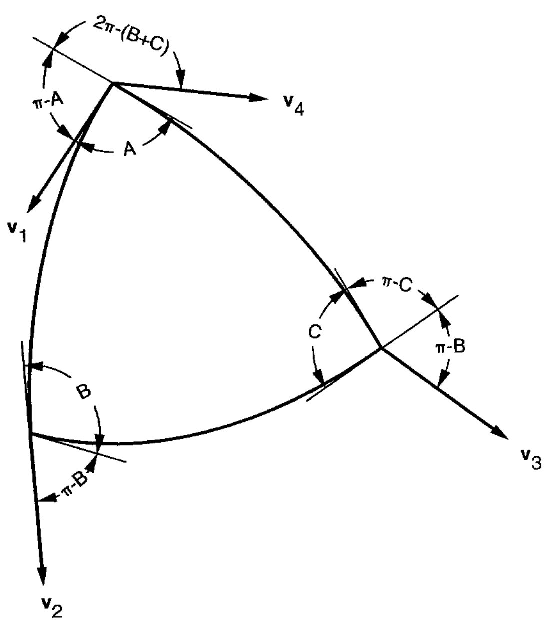


Fig. 4.4. The vector v_1 is parallel transported along the great circle arc AB to v_2, then along the great circle arc BC where it becomes v_3. Finally it is parallel transported along great circle CA where it becomes v_4. Careful computation reveals that v_1 in the guise of v_4 has undergone a rotation of magnitude $A + B + C - \pi$.

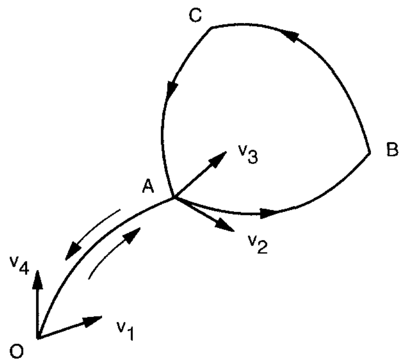


Fig. 4.5. When v_1 is parallel transported around the path OABCAO it becomes v_2, then v_3, and then finally v_4. The angle between v_1 and v_4 is the same as the angle between v_2 and v_3.

triangle then:

$$\begin{aligned}\alpha &= (1/r^2)(\text{Area of } \triangle\text{ABC}) \\ &= (\text{Gaussian curvature of sphere})(\text{Area of } \triangle\text{ABC}). \qquad (4.2)\end{aligned}$$

This last result can be generalized in several ways. It is possible to show that for a fairly broad class of closed paths, the angle of rotation for a vector parallel transported around a member of that class is equal to $(1/r^2)$ multiplied by the area enclosed by that path. Consider Fig. 4.5. Suppose all arc segments are great circles and the closed path is OABCAO. Suppose vector v_1 parallel transported along OA becomes v_2 at A and then v_3 at A after it is also parallel transported around the spherical triangle ABC. Then finally suppose v_3 is parallel transported into v_4 when it is carried along the last leg of our closed path from A to O. It is not difficult to see that the angle between v_1 and v_4 is the same as the angle between v_2 and v_3. To get that result, we merely note that on the last leg of our journey from A to O not only does v_3 parallel transport into v_4 but v_2 parallel transports into v_1. For this reason the angle of rotation for the closed path OABCAO is the same as the angle of rotation for a vector simply parallel transported about the perimeter of the spherical triangle.

It is not too difficult to generalize further. For example, if the region encompassed by the closed path can be decomposed into a finite sum of spherical triangles then the angle of rotation caused by parallel transport of a vector around the closed path will still equal $(1/r^2)$ multiplied by the area surrounded by the closed path. (See Problems 4.1 and 4.2.)

One can also define the notion of parallel transport of vectors along paths which are not geodesics. If the path is reasonably well behaved then one may consider it to be a limit of a sequence of polygonal paths consisting of geodesic segments. One could determine the parallel transport of a vector on each of the polygonal paths in the sequence and then take some

appropriated limit. However, this approach leads to massive computations which I would like to avoid. For this reason, we will return to this problem with a different approach in Section 4.3 of this chapter.

Meanwhile, we have created enough machinery to consider the curvature of 2-dimensional surfaces in E^3 which are not necessarily spherical. For a sphere, the Gaussian curvature is $(1/r^2)$ and is constant. More generally, a 2-dimensional surface has a Gaussian curvature which varies from point to point. One way of defining Gaussian curvature is to make use of the notion of parallel transport of a vector in much the same fashion that it has been just discussed. If the surface is smooth near a given point x, then we can choose two other points y and z such that there exist three geodesic segments forming a *geodesic triangle* with non-zero area. One of the three geodesic segments would have end points x and y, another end points y and z, and of course the third segment would have end points z and x. If $K(x)$ is used to designate the *Gaussian curvature* at x, then

$$K(x) = \lim_{d \to 0} \alpha/(\text{Area of geodesic triangle}), \tag{4.3}$$

where x is one of the vertices of the geodesic triangle, d is the length of the longest edge, and α is the angle of rotation for a vector parallel transported about the perimeter of the geodesic triangle.

Let us apply this definition of Gaussian curvature to a couple of examples. The most trivial example is a flat surface. Consider a flat sheet of paper. On such a sheet of paper geodesics are straight lines which can be constructed with a pencil and ruler. In this case a "geodesic triangle" is an ordinary triangle. When we parallel transport a vector about such a triangle, the result is a zero rotation. Thus by the definition that appears in Eq. (4.3), we get a zero value for the Gaussian curvature. This is indeed no great surprise.

What is somewhat less obvious is the fact that if the paper is bent without stretching, the resulting surface still has a zero value for its Gaussian curvature at all points. If for example the paper is bent into a cylinder or a cone, the edges of a penciled triangle are no longer straight lines in 3-space. Nevertheless they are still geodesics in the surface of the paper. The angles at the vertices remain the same and therefore the parallel transport of a vector about the triangle would still result in a zero rotation. Using Eq. (4.3), we would still get a zero value for the Gaussian curvature.

An observer who is constrained to take all measurements in the surface of the paper would not be able to distinguish between the flat sheet of paper and the bent sheet of paper. Perhaps we should insert a condition or proviso in this last statement. If the paper were infinite and it was bent into the shape of an infinite sine wave, then the statement would indeed be true. (See Fig. 4.6.) On the other hand, if the paper were rolled into a cylinder or cone there would be some geodesics along which the observer could travel in one direction and eventually return to some point encountered previously on the

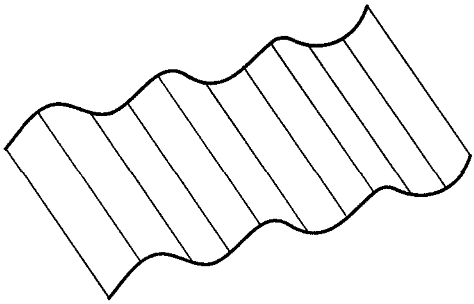


Fig. 4.6. A piece of paper which is bent without stretching would be locally indistinguishable from a flat surface to an observer constrained to take all measurements in the surface of the paper.

trip. In this situation the observer could safely conclude that we were not dealing with a flat sheet of paper. However, if we require the observer to take all measurements not only in the surface but also in some sufficiently small local region then it would indeed be impossible for the observer to determine whether or not the local region was flat or bent in 3-space.

In this chapter, we have so far considered the sphere which has positive Gaussian curvature and some other 2-dimensional surfaces that have zero Gaussian curvature. On the other hand there are surfaces with a negative Gaussian curvature. (See Fig. 4.7.)

On the spherical surface of the earth, two meridians which appear parallel at the equator converge and intersect at the North and South Poles. By contrast, on a surface of negative curvature, geodesics which appear parallel at one location diverge from one another. (See Figs. 4.7 and 4.8.) As a result triangles formed from geodesics have the property that the sum of their angles is less than π. (See Fig. 4.9.) As a result a vector which is parallel transported around the perimeter in the counterclockwise direction will be rotated in the negative or clockwise direction.

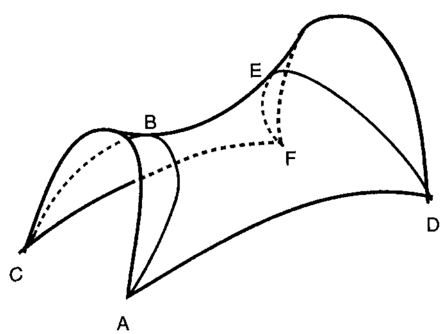


Fig. 4.7. On a surface of negative curvature, geodesics which appear parallel at one location diverge from one another

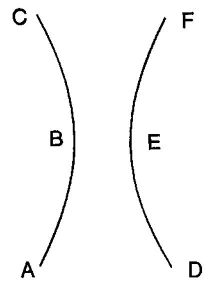


Fig. 4.8. Diverging geodesics from Fig. 4.7.

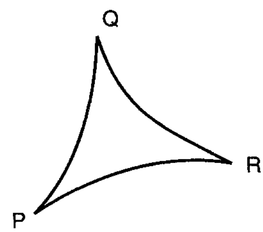


Fig. 4.9. A triangle formed from geodesics on a surface of negative curvature. The sum of the three angles is less than π.

Problem 4.1. Consider Fig. 4.10. Suppose ABC and DEF are spherical triangles. Convince yourself that the angle of rotation caused by the parallel transport of a vector along the path OABCAODEFCBDO is the same as the angle of rotation caused by the parallel transport of a vector along path OABDEFCAO.

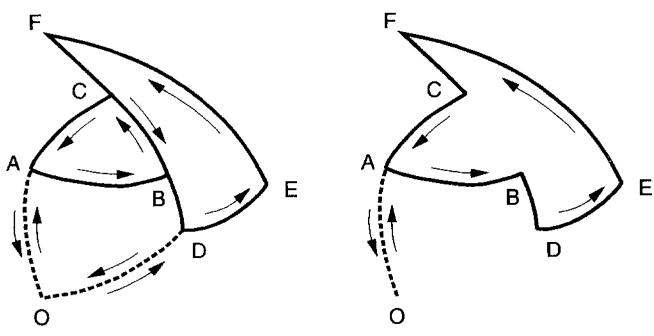


Fig. 4.10. On a spherical surface, the amount of rotation caused by parallel transport along the path OABCAODEFCBDO is the same as the rotation caused by parallel transport along the path OABDEFCAO. In both cases, the amount of rotation is $(1/r^2) \times$ (area of figure bounded by the curve ABDEFCA).

Problem 4.2. Consider a closed path on the surface of a sphere which is composed of a finite number of segments of geodesics. Suppose also that the region encompassed by the closed path may be decomposed into a finite sum of spherical triangles. Use the result of Problem 4.1. to convince yourself that the angle of rotation caused by the parallel transport of a vector around such a closed path is equal to $(1/r^2)$ multiplied by the area encompassed by the closed path.

4.2 The Operator ∇_v on an m-Dimensional Surface Embedded in an n-Dimensional Flat Space

Consider an n-dimensional Euclidean or pseudo-Euclidean space. In such a space we have a position vector s which can be used to derive all tangent vectors

$$s = \sum_{k=1}^{n} x^k \hat{\gamma}_k. \tag{4.4}$$

As in Chapter 3, it is understood that the $\hat{\gamma}_k$'s are orthonormal. That is

$$(\hat{\gamma}_k)^2 = \pm I \;\forall k \quad \text{and} \quad \hat{\gamma}_k\hat{\gamma}_j + \hat{\gamma}_j\hat{\gamma}_k = 0 \quad \text{if } j \neq k.$$

Now suppose we consider an alternate coordinate system $u^1, u^2, \ldots, u^n$. Then

$$\begin{aligned} x^1 &= x^1(u^1, u^2, \ldots, u^n) \\ x^2 &= x^2(u^1, u^2, \ldots, u^n) \\ &\;\vdots \\ x^n &= x^n(u^1, u^2, \ldots, u^n). \end{aligned} \tag{4.5}$$

From this last set of equations, we can obtain the lower index Dirac matrices corresponding to the u^α coordinate system:

$$\gamma_\alpha = \frac{\partial s}{\partial u^\alpha} = \sum_{k=1}^{n} \left(\frac{\partial x^k}{\partial u^\alpha}\right)\hat{\gamma}_k. \tag{4.6}$$

(In this section, we will use barred entities when we discuss the full n-dimensional space and unbarred entities when we discuss lower dimensional surfaces. Since the range of summation of the indices will vary, we will also add the summation sign in equations where the range of summation may not be obvious.)

The corresponding contravariant Dirac matrices will be determined by computing the inverse matrix of the metric tensor $\bar{g}_{\alpha\beta}$ and then using the

resulting matrix $\bar{g}^{\alpha\beta}$ as an index raising operator. In this fashion, we obtain the equation:

$$\bar{\gamma}^{\alpha} = \sum_{\beta=1}^{n} \bar{g}^{\alpha\beta}\bar{\gamma}_{\beta}. \tag{4.7}$$

For the example of spherical coordinates in E^3,

$$s = \sum_{k=1}^{3} x^k \hat{\gamma}_k = r \sin\theta \cos\phi \hat{\gamma}_1 + r \sin\theta \sin\phi \hat{\gamma}_2 + r \cos\theta \hat{\gamma}_3, \tag{4.8}$$

$$\bar{\gamma}_r = \frac{\partial s}{\partial r} = \sin\theta \cos\phi \hat{\gamma}_1 + \sin\theta \sin\phi \hat{\gamma}_2 + \cos\theta \hat{\gamma}_3, \tag{4.9}$$

$$\bar{\gamma}_\theta = \frac{\partial s}{\partial \theta} = r \cos\theta \cos\phi \hat{\gamma}_1 + r \cos\theta \sin\phi \hat{\gamma}_2 - r \sin\theta \hat{\gamma}_3, \tag{4.10}$$

and

$$\bar{\gamma}_\phi = \frac{\partial s}{\partial \phi} = -r \sin\theta \sin\phi \hat{\gamma}_1 + r \sin\theta \cos\phi \hat{\gamma}_2. \tag{4.11}$$

The metric tensor is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2\theta \end{bmatrix}$$

and the inverse matrix is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/r^2 & 0 \\ 0 & 0 & 1/(r^2 \sin^2\theta) \end{bmatrix}.$$

Therefore from Eq. (4.7), we have

$$\bar{\gamma}^r = \bar{\gamma}_r, \qquad \bar{\gamma}^\theta = (1/r^2)\bar{\gamma}_\theta, \qquad \text{and} \qquad \bar{\gamma}^\phi = (1/(r^2 \sin^2\theta))\bar{\gamma}_\phi. \tag{4.12}$$

In the general case, a differential tangent vector at a given point can be written as

$$ds = \sum_{\alpha=1}^{n} \bar{\gamma}_\alpha \, du^\alpha. \tag{4.13}$$

Given any coordinate system, we can construct an m-dimensional surface

by setting the last $n-m$ coordinates equal to some appropriate set of constants. (To be perverse, we will refer to the example of the sphere in which it is the first coordinate "r" that is set equal to some constant.) The points on such an m-dimensional surface are parameterized by the first m coordinates. That is

$$\begin{aligned} x^1 &= x^1(u^1, u^2, \ldots, u^m, c^{m+1}, c^{m+2}, \ldots, c^n), \\ x^2 &= x^2(u^1, u^2, \ldots, u^m, c^{m+1}, c^{m+2}, \ldots, c^n), \\ &\vdots \\ x^n &= x^n(u^1, u^2, \ldots, u^m, c^{m+1}, c^{m+2}, \ldots, c^n). \end{aligned} \tag{4.14}$$

When one parameterizes a given surface in this way or any other way, one generally encounters difficulties. One cannot usually hope to construct a single set of functions which will parameterize an entire surface and be well behaved at all points. For example, in the usual parameterization of the sphere $\gamma_\phi = 0$ at the poles where $\theta = 0$ or π. In general a single set of parameterizing functions is useful only in some local area. To treat an entire surface, one must usually patch together several parameterizations on overlapping areas. In what follows, we discuss only one parameterization on a given surface so the results apply only to a local region of the surface.

A *regular point* on our m-dimensional surface is a point at which the coordinate system of Dirac matrices is linearly independent. At any regular point on our m-dimensional surface, there is an m-dimensional plane which is tangent to the surface at that point. Furthermore, if the given point is considered to be the origin of the plane, then this tangent plane is spanned by the first m $\bar{\gamma}_\alpha$'s. In this context, we have an m-dimensional space of tangent vectors which is indeed tangent to something.

In spherical coordinates, every point on a sphere is regular except the two poles. At any regular point, the tangent plane is spanned by γ_θ and γ_ϕ. (See Fig. 4.11.)

In more general cases, a differential tangent vector in our m-dimensional surface may be written in the form:

$$\mathrm{d}s = \sum_{\alpha=1}^{m} \bar{\gamma}_\alpha \, \mathrm{d}x^\alpha. \tag{4.15}$$

If we designate the lower index Dirac matrices in the new space by γ_α for $\alpha = 1, 2, \ldots, m$, then

$$\gamma_\alpha = \bar{\gamma}_\alpha \quad \text{for } \alpha = 1, 2, \ldots, m. \tag{4.16}$$

Furthermore, since $g_{\alpha\beta} I = \frac{1}{2}(\gamma_\alpha\gamma_\beta + \gamma_\beta\gamma_\alpha)$, we can obtain the new metric tensor by simply suppressing the indices of $\bar{g}_{\alpha\beta}$ which are greater than m.

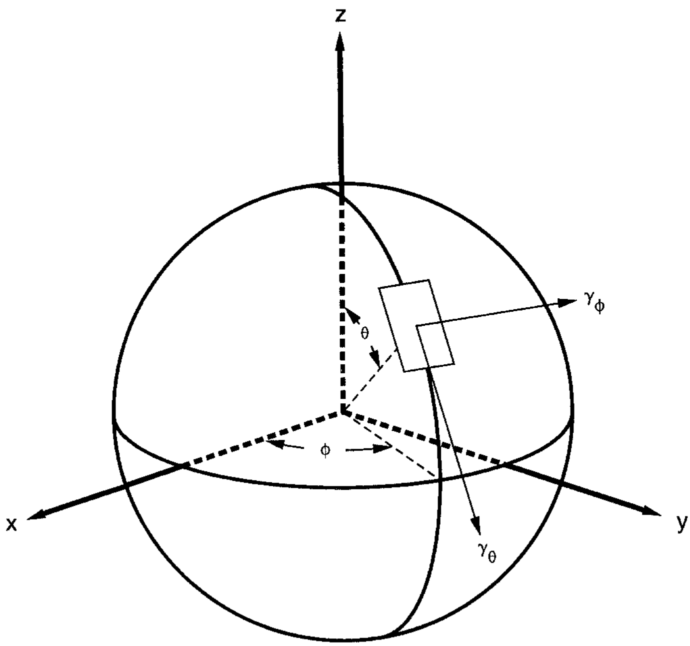


Fig. 4.11. The plane tangent to the sphere is spanned by γ_θ and γ_ϕ.

That is,

$$g_{\alpha\beta} = \bar{g}_{\alpha\beta} \quad \text{for } \alpha, \beta = 1, 2, \ldots, m. \tag{4.17}$$

The computation of the new upper index Dirac matrices is not as simple. Although the complete set of n upper index $\bar{\gamma}^\alpha$'s spans the same space as the complete set of n lower index $\bar{\gamma}_\alpha$'s, it is not necessarily true that the first m $\bar{\gamma}^\alpha$'s span the same subspace as the first m $\bar{\gamma}_\alpha$'s. To determine the new set of upper index Dirac matrices, one must first compute the inverse matrix of the new metric tensor $g_{\alpha\beta}$. We can then obtain the new upper index Dirac matrices by the relation

$$\gamma^\alpha = \sum_{\beta=1}^{m} g^{\alpha\beta}\gamma_\beta \quad \text{for } \alpha = 1, 2, \ldots, m. \tag{4.18}$$

To illustrate some of these points, let us consider an example of a 2-dimensional plane in E^3. In E^3, the position vector s can be written in the form

$$s = x^1\hat{\gamma}_1 + x^2\hat{\gamma}_2 + x^3\hat{\gamma}_3.$$

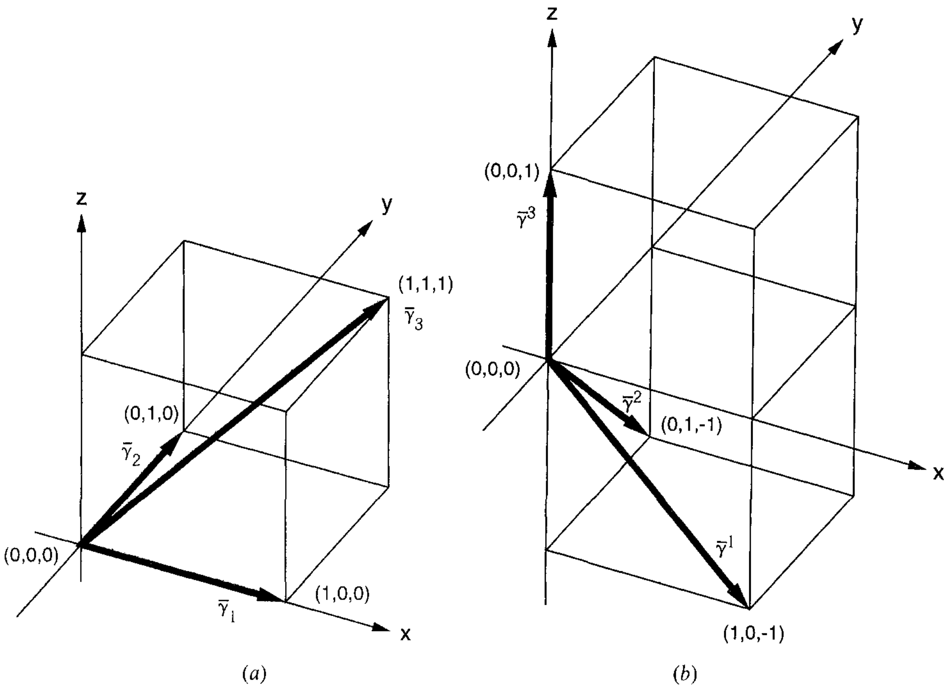


Fig. 4.12. (*a*) The tangent vectors $\bar{\gamma}_1$, $\bar{\gamma}_2$, and $\bar{\gamma}_3$ form a coordinate system of lower index Dirac matrices for the set of coordinates $\{u^1, u^2, u^3\}$ where $x^1 = u^1 + u^2$, $x^2 = u^2 + u^3$, and $x^3 = u^3$ (*b*) The upper index Dirac matrices $\bar{\gamma}^1$, $\bar{\gamma}^2$, and $\bar{\gamma}^3$ are *biorthogonal* to the system of lower index matrices shown in (*a*), that is, the scalar product $\langle \bar{\gamma}_\alpha, \bar{\gamma}^\beta \rangle = \delta_\alpha^\beta$.

Suppose we introduce an alternate set of coordinates $\{u^1, u^2, u^3\}$, where

$$x^1 = u^1 + u^3, \qquad x^2 = u^2 + u^3, \qquad \text{and} \qquad x^3 = u^3.$$

In this alternate coordinate system,

$$s = (u^1 + u^3)\hat{\gamma}_1 + (u^2 + u^3)\hat{\gamma}_2 + u^3\hat{\gamma}_3$$

and

$$\mathrm{d}s = \hat{\gamma}_1 \,\mathrm{d}u^1 + \hat{\gamma}_2 \,\mathrm{d}u^2 + (\hat{\gamma}_1 + \hat{\gamma}_2 + \hat{\gamma}_3)\,\mathrm{d}u^3.$$

Since $\bar{\gamma}_\alpha = \partial s/\partial u^\alpha$, the members of the coordinate system of lower index Dirac matrices are

$$\bar{\gamma}_1 = \hat{\gamma}_1, \qquad \bar{\gamma}_2 = \hat{\gamma}_2, \qquad \text{and} \qquad \bar{\gamma}_3 = \hat{\gamma}_1 + \hat{\gamma}_2 + \hat{\gamma}_3. \tag{4.19}$$

(This system of lower index vectors is illustrated in Fig. 4.12*a*.)

Using the relation that $\bar{g}_{\alpha\beta}I = \frac{1}{2}(\bar{\gamma}_\alpha\bar{\gamma}_\beta + \bar{\gamma}_\beta\bar{\gamma}_\alpha)$, we find that

$$(\bar{g}_{\alpha\beta}) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 3 \end{bmatrix}. \tag{4.20}$$

Computing the inverse of $\bar{g}_{\alpha\beta}$, we get

$$(\bar{g}^{\alpha\beta}) = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 2 & -1 \\ -1 & -1 & 1 \end{bmatrix}. \tag{4.21}$$

Using Eqs. (4.21) and (4.19), we get

$$\begin{aligned} \bar{\gamma}^1 &= 2\bar{\gamma}_1 + \bar{\gamma}_2 - \bar{\gamma}_3 = \hat{\gamma}_1 - \hat{\gamma}_3 \\ \bar{\gamma}^2 &= \bar{\gamma}_1 + 2\bar{\gamma}_2 - \bar{\gamma}_3 = \hat{\gamma}_2 - \hat{\gamma}_3 \\ \bar{\gamma}^3 &= -\bar{\gamma}_1 - \bar{\gamma}_2 + \bar{\gamma}_3 = \hat{\gamma}_3. \end{aligned} \tag{4.22}$$

We note that the lower index and upper index Dirac matrices form a biorthonormal system of vectors. That is,

$$\langle \bar{\gamma}_\alpha, \bar{\gamma}^\beta \rangle = \delta_\alpha^\beta. \tag{4.23}$$

Equation (4.23) may be checked algebraically using Eqs. (4.19) and (4.22). Alternatively, we can obtain a pictorial image of the situation by carefully comparing Fig. 4.12*a* with Fig. 4.12*b*. We note, for example, that $\bar{\gamma}^1$ is perpendicular to the plane spanned by $\bar{\gamma}_2$ and $\bar{\gamma}_3$.

If we now discuss the surface defined by $u^3 = 0$, we are in effect talking about the xy-plane. At each point on this plane, the tangent space is spanned by γ_1 and γ_2, where

$$\gamma_1 = \bar{\gamma}_1 = \hat{\gamma}_1 \qquad \text{and} \qquad \gamma_2 = \bar{\gamma}_2 = \hat{\gamma}_2. \tag{4.24}$$

Examining Fig. 4.12*b*, we see that neither $\bar{\gamma}^1$ nor $\bar{\gamma}^2$ lie in the tangent space at any point on the xy-plane. To compute x^1 and x^2, we first recognize the fact that our new metric tensor is simply the 2×2 identity matrix, that is

$$(g_{\alpha\beta}) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Thus the inverse matrix $(g^{\alpha\beta})$ is also the 2×2 identity matrix. Therefore,

from Eq. (4.18), we have

$$\gamma^1 = \gamma_1 = \hat{\gamma}_1 \qquad \text{and} \qquad \gamma^2 = \gamma_2 = \hat{\gamma}_2. \tag{4.25}$$

Another example worth considering is the surface of a sphere. Using the usual spherical coordinate system, the metric tensor $g_{\alpha\beta}$ is diagonal. For this situation, we can obtain the elements of the contravariant tensor $g^{\alpha\beta}$ by simply suppressing the same indices that are suppressed to obtain $g_{\alpha\beta}$ from $\bar{g}_{\alpha\beta}$. In particular, for the 2-dimensional surface of the sphere embedded in E^3, we have

$$\gamma_\theta = \bar{\gamma}_\theta \qquad \text{and} \qquad \gamma_\phi = \bar{\gamma}_\phi. \tag{4.26}$$

The metric tensor for the curved surface written in matrix form is

$$(g_{\alpha\beta}) = \begin{bmatrix} r^2 & 0 \\ 0 & r^2 \sin^2\theta \end{bmatrix}. \tag{4.27}$$

The inverse matrix is

$$(g^{\alpha\beta}) = \begin{bmatrix} 1/r^2 & 0 \\ 0 & 1/(r^2 \sin^2\theta) \end{bmatrix}, \tag{4.28}$$

so

$$\gamma^\theta = (1/r^2)\gamma_\theta = \bar{\gamma}^\theta \qquad \text{and} \qquad \gamma^\phi = (1/(r^2\sin^2\theta))\gamma_\phi = \bar{\gamma}^\phi. \tag{4.29}$$

A member of a coordinate system of Dirac matrices, when considered as a tangent vector, can have different directions and different lengths at different points in space. For example, on the surface of the globe, γ_θ has the direction that is labeled "South". However the direction "South" for a resident of Greenland is quite different from the direction "South" for a resident of Japan. Furthermore γ_ϕ, which has the direction "East" at all points of the globe, has a magnitude of $r \sin\theta$ where θ is the angle from the North Pole as shown in Fig. 4.11. Thus we see that these tangent vectors are functions of the points at which they are evaluated.

When a tangent vector is evaluated on a sequence of points along some path, we can consider it to be undergoing a change as it is moved along that path. In this context, it makes sense to ask how fast does the tangent vector change or what are its derivatives. From Eqs. (4.9), (4.10), and (4.11), we find that

$$\frac{\partial\bar{\gamma}_\theta}{\partial\theta} = -r\sin\theta\cos\phi\hat{\gamma}_1 - r\sin\theta\sin\phi\hat{\gamma}_2 - r\cos\theta\hat{\gamma}_3$$

or

$$\frac{\partial \bar{\gamma}_\theta}{\partial \theta} = -r\bar{\gamma}_r. \tag{4.30}$$

In a similar fashion, we can compute

$$\frac{\partial \bar{\gamma}_\theta}{\partial \phi} = \frac{\cos\theta}{\sin\theta}\bar{\gamma}_\phi \tag{4.31}$$

$$\frac{\partial \bar{\gamma}_\phi}{\partial \theta} = \frac{\cos\theta}{\sin\theta}\bar{\gamma}_\phi \tag{4.32}$$

$$\frac{\partial \bar{\gamma}_\phi}{\partial \phi} = -r\sin^2\theta\bar{\gamma}_r - \sin\theta\cos\theta\bar{\gamma}_\theta. \tag{4.33}$$

These quantities represent the rates of change as seen by an observer who is permitted to take measurements in all three dimensions. However, an observer who is constrained by take measurements on the surface of the sphere would not observe any changes in the $\bar{\gamma}_r$ direction. Such an observer would detect only the changes in the $\bar{\gamma}_\theta$ or $\bar{\gamma}_\phi$ direction.

Therefore if we wish to compute the rate of change of γ_θ or γ_ϕ on the surface of the sphere as seen by our 2-dimensional observer then we must suppress the γ_r component. Thus, in place of $\partial/\partial\theta$ and $\partial/\partial\phi$, it becomes necessary to define two new operators ∇_θ and ∇_ϕ. Suppressing the $\bar{\gamma}_r$ component in Eqs. (4.30), (4.31), (4.32), and (4.33), we get

$$\nabla_\theta\gamma_\theta = 0 \tag{4.34}$$

$$\nabla_\phi\gamma_\theta = \frac{\cos\theta}{\sin\theta}\gamma_\phi \tag{4.35}$$

$$\nabla_\theta\gamma_\phi = \frac{\cos\theta}{\sin\theta}\gamma_\phi \tag{4.36}$$

$$\nabla_\phi\gamma_\phi = -\sin\theta\cos\theta\gamma_\theta. \tag{4.37}$$

In the general case, the situation is somewhat similar. In the context of the entire space $\partial\gamma_\alpha/\partial u^\beta$ can be expanded as a linear combination of the $\bar{\gamma}_\alpha$'s. Therefore we may write

$$\partial\gamma_\alpha/\partial u^\beta = \sum_{\nu=1}^{n} \bar{\Gamma}_{\beta\alpha}{}^{\nu}\bar{\gamma}_\nu. \tag{4.38}$$

However, in the general case, when we wish to compute the corresponding

rates of change observed by an m-dimensional observer in an m-dimensional surface, we cannot simply suppress the last $n - m$ indices. The reason for this is that if we do not start out with an orthogonal coordinate system, then the space spanned by the last $n - m$ $\bar{\gamma}_\alpha$'s may not be orthogonal to the space spanned by the first m $\bar{\gamma}_\alpha$'s. What we wish to do is to project $\partial\gamma_\alpha/\partial u^\beta$ onto the subspace spanned by the first m $\bar{\gamma}_\alpha$'s. To do this we would like to decompose $\partial\gamma_\alpha/\partial u^\beta$ into a sum of two tangent vectors, one of which is in the subspace spanned by the first m $\bar{\gamma}_\alpha$'s and the other is in the orthogonal complement. The orthogonal complement is spanned by the last $n - m$ contravariant $\bar{\gamma}^\alpha$'s. Thus when this decomposition is possible, we may write

$$\frac{\partial\gamma_\alpha}{\partial u^\beta} = \sum_{\nu=1}^{m} \Gamma_{\beta\alpha}{}^{\nu}\gamma_\nu + \sum_{\nu=m+1}^{n} A_{\beta\alpha\nu}\bar{\gamma}^\nu. \tag{4.39}$$

Such a decomposition is not always possible. (See Problem 4.3.) In a pseudo Euclidean space, there are vectors which are orthogonal to themselves. For example, suppose $\hat{\gamma}_0$ and $\hat{\gamma}_1$ are members of an orthonormal basis where $(\hat{\gamma}_0)^2 = -(\hat{\gamma}_1)^2 = I$. If $\boldsymbol{v} = \hat{\gamma}_0 + \hat{\gamma}_1$, then $\langle \boldsymbol{v}, \boldsymbol{v}\rangle = 0$. Such vectors have been variously referred to as *isotropic*, *null*, or *light-like*.

Given an m-dimensional subspace, the orthogonal complement is a subspace of dimension $n - m$. However, the m-dimensional subspace may contain vectors which are not only orthogonal to themselves but to every other vector in the subspace. When this is true, these same vectors are also contained in the orthogonal complement. In this situation, the m-dimensional subspace together with the $(n - m)$-dimensional orthogonal complement do not span the entire space.

We will avoid opening this Pandora's box by restricting ourselves to the case for which the metric tensor for the m-dimensional subspace has a non-zero determinant. (See Problem 4.5.)

When the determinant of $(g_{\alpha\beta})$ is zero, the Clifford algebra that accompanies the m-dimensional plane is said to be *singular*. The nature of singular Clifford algebras is discussed at length in a set of mimeographed notes by Marcel Riesz (1958).

To compute the $\Gamma_{\alpha\beta}{}^{\nu}$'s from Eq. (4.39), we can take advantage of the fact that the γ^β's (not the $\bar{\gamma}^\beta$'s) span the same subspace as the first m $\bar{\gamma}_\alpha$'s. Thus

$$\begin{aligned}\langle \partial\gamma_\beta/\partial u^\alpha, \gamma^\nu\rangle &= \sum_{\eta=1}^{m} \Gamma_{\alpha\beta}{}^{\eta}\langle\gamma_\eta, \gamma^\nu\rangle + \sum_{\eta=m+1}^{n} A_{\alpha\beta\eta}\langle\bar{\gamma}^\eta, \gamma^\nu\rangle \\ &= \Gamma_{\alpha\beta}{}^{\eta}\,\delta_\eta^\nu\end{aligned}$$

and therefore

$$\Gamma_{\alpha\beta}{}^{\nu} = \langle \partial\gamma_\beta/\partial u^\alpha, \gamma^\nu\rangle. \tag{4.40}$$

Now if we project out that component of $\partial\gamma_\beta/\partial u^\alpha$ which is detectable to

the m-dimensional observer, we get

$$\nabla_\alpha \gamma_\beta = \sum_{\nu=1}^{m} \Gamma_{\alpha\beta}{}^{\nu}\gamma_\nu \quad \text{for } \alpha, \beta = 1, 2, \ldots, m. \tag{4.41}$$

The $\Gamma_{\alpha\beta}{}^{\nu}$'s are known as *Christoffel symbols*. Christoffel symbols are not tensors because they do not have the required transformation properties under changes in the coordinate system. This point will be developed further in the next chapter.

In most presentations the formula used for Christoffel symbols is quite different from Eq. (4.40). The usual formula is

$$\Gamma_{\alpha\beta}{}^{\nu} = \frac{g^{\nu\eta}}{2}\left(\frac{\partial g_{\eta\alpha}}{\partial u^\beta} + \frac{\partial g_{\eta\beta}}{\partial u^\alpha} - \frac{\partial g_{\alpha\beta}}{\partial u^\eta}\right). \tag{4.42}$$

To derive Eq. (4.42) from Eq. (4.40), we first observe that

$$\frac{\partial \gamma_\beta}{\partial u^\alpha} = \frac{\partial^2 s}{\partial u^\alpha \, \partial u^\beta} = \frac{\partial \gamma_\alpha}{\partial u^\beta}. \tag{4.43}$$

Also $\gamma^\nu = g^{\nu\eta}\gamma_\eta$. With these thoughts in mind, we have from Eq. (4.40)

$$\Gamma_{\alpha\beta}{}^{\nu} = g^{\nu\eta}\left\langle \frac{\partial \gamma_\beta}{\partial u^\alpha}, \gamma_\eta \right\rangle = \tfrac{1}{2} g^{\nu\eta}\left(\left\langle \frac{\partial \gamma_\beta}{\partial u^\alpha}, \gamma_\eta \right\rangle + \left\langle \frac{\partial \gamma_\alpha}{\partial u^\beta}, \gamma_\eta \right\rangle\right).$$

Since

$$\left\langle \frac{\partial \gamma_\sigma}{\partial u^\lambda}, \gamma_\eta \right\rangle = \frac{\partial}{\partial u^\lambda}\langle \gamma_\sigma, \gamma_\eta \rangle - \left\langle \gamma_\sigma, \frac{\partial \gamma_\eta}{\partial u^\lambda} \right\rangle$$

for arbitrary σ and λ, we see that

$$\Gamma_{\alpha\beta}{}^{\nu} = \frac{g^{\nu\eta}}{2}\left(\frac{\partial}{\partial u^\alpha}\langle \gamma_\beta, \gamma_\eta \rangle - \left\langle \gamma_\beta, \frac{\partial \gamma_\eta}{\partial u^\alpha} \right\rangle + \frac{\partial}{\partial u^\beta}\langle \gamma_\alpha, \gamma_\eta \rangle - \left\langle \gamma_\alpha, \frac{\partial \gamma_\eta}{\partial u^\beta} \right\rangle\right).$$

Using the relation that $\langle \gamma_\alpha, \gamma_\beta \rangle = g_{\alpha\beta}$ and repeating the use of Eq. (4.43) with a different set of indices, it immediately follows that

$$\Gamma_{\alpha\beta}{}^{\nu} = \frac{g^{\nu\eta}}{2}\left(\frac{\partial g_{\beta\eta}}{\partial u^\alpha} + \frac{\partial g_{\alpha\eta}}{\partial u^\beta} - \left\langle \gamma_\beta, \frac{\partial \gamma_\alpha}{\partial u^\eta} \right\rangle - \left\langle \gamma_\alpha, \frac{\partial \gamma_\beta}{\partial u^\eta} \right\rangle\right)$$

and

$$\Gamma_{\alpha\beta}{}^{\nu} = \frac{g^{\nu\eta}}{2}\left(\frac{\partial g_{\beta\eta}}{\partial u^\alpha} + \frac{\partial g_{\alpha\eta}}{\partial u^\beta} - \frac{\partial g_{\alpha\beta}}{\partial u^\eta}\right).$$

The fact that the Christoffel symbols can be written in terms of the elements of the metric tensor for the m-dimensional surface serves to underline the point that $\nabla_\alpha \gamma_\beta$ is an entity which is measurable by our m-dimensional observer.

Reviewing our definitions, $\nabla_\alpha \gamma_\beta$ represents the orthogonal projection of $\partial \gamma_\beta / \partial u^\alpha$ or $\partial^2 s / \partial u^\alpha \, \partial u^\beta$ onto the m-dimensional plane which is tangent to our m-dimensional surface at some given point. Since the same thing can be said for $\nabla_\beta \gamma_\alpha$, we can write

$$\nabla_\alpha \gamma_\beta = \nabla_\beta \gamma_\alpha. \tag{4.44}$$

For purposes of future generalization in the next chapter, Eq. (4.44) is an important relation. Meanwhile, we note that an immediate consequence of Eq. (4.42) or alternatively Eqs. (4.41) and (4.44) is the symmetry of the Christoffel symbols in the lower indices, that is

$$\Gamma_{\alpha\beta}{}^{\nu} = \Gamma_{\beta\alpha}{}^{\nu}. \tag{4.45}$$

The operator ∇_η can be applied to entities other than the γ_α's. Suppose we consider a Clifford number of the form

$$\mathscr{F} = F^{\nu_1 \nu_2, \ldots, \nu_p}(u^1, u^2, \ldots, u^m) \gamma_{\nu_1} \gamma_{\nu_2} \cdots \gamma_{\nu_p}.$$

Then we define $\nabla_\eta \mathscr{F}$ as

$$\begin{aligned} \nabla_\eta \mathscr{F} &= (\nabla_\eta F^{\nu_1 \nu_2 \ldots \nu_p}) \gamma_{\nu_1} \gamma_{\nu_2} \cdots \gamma_{\nu_p} + F^{\nu_1 \nu_2 \ldots \nu_p}) (\nabla_\eta \gamma_{\nu_1}) \gamma_{\nu_2} \cdots \gamma_{\nu_p} \\ &\quad + F^{\nu_1 \nu_2 \ldots \nu_p} \gamma_{\nu_1} (\nabla_\eta \gamma_{\nu_2}) \gamma_{\nu_3} \cdots \gamma_{\nu_p} + \ldots + F^{\nu_1 \nu_2 \ldots \nu_p} \gamma_{\nu_1} \gamma_{\nu_2} \cdots (\nabla_\eta \gamma_{\nu_p}). \end{aligned} \tag{4.46}$$

In Eq. (4.46), it is understood that

$$\nabla_\eta F^{\nu_1 \nu_2 \ldots \nu_p} = \frac{\partial}{\partial u^\eta} F^{\nu_1 \nu_2 \ldots \nu_p}. \tag{4.47}$$

The definition of $\nabla_\eta \mathscr{F}$ that appears in Eq. (4.46) may be regarded as the projection of $\partial \mathscr{F} / \partial u^\eta$ onto the space of Clifford numbers associated with a given point on the m-dimensional surface. (See Problem 4.4.)

The operator ∇_η behaves very much like the operator $\partial / \partial u^\eta$. However, there is one major exception. Unless the m-dimensional surface is intrinsically indistinguishable from a flat subspace, $\nabla_\alpha \nabla_\beta \neq \nabla_\beta \nabla_\alpha$. In particular

$$(\nabla_\alpha \nabla_\beta - \nabla_\beta \nabla_\alpha) \gamma_\nu = R_{\alpha\beta\nu}{}^{\eta} \gamma_\eta. \tag{4.48}$$

where $R^{\nu}{}_{\nu\alpha\beta}$ is known as the *Riemann curvature tensor*. As one might suspect from the name, this tensor is a measure of the curvature of the m-dimensional

surface at any given point on that surface. This point will be developed further in the next chapter.

Problem 4.3. Consider Minkowski 4-space where $(\hat{\gamma}_1)^2 = (\hat{\gamma}_2)^2 = (\hat{\gamma}_3)^2 = -I$ and $(\hat{\gamma}_0)^2 = -I$. Consider the 1-dimensional subspace spanned by $\hat{\gamma}_0 + \hat{\gamma}_1$. Show that the orthogonal complement is 3-dimensional. Show that the vector $\hat{\gamma}_0 - \hat{\gamma}_1$ cannot be decomposed into a multiple of $\hat{\gamma}_0 + \hat{\gamma}_1$ and a vector in the orthogonal complement.

Problem 4.4. This last section was devoted to the construction of tools for the study of m-dimensional surfaces embedded in an n-dimensional Euclidean or pseudo-Euclidean space. At each point on the m-dimensional surface, we have an m-dimensional plane or hyperplane spanned by $\{\gamma_1, \gamma_2, \ldots, \gamma_m\}$ or $\{\gamma^1, \gamma^2, \ldots, \gamma^m\}$. When that same point is considered as a member of the larger n-dimensional space, we may identify with that point an orthogonal complement space of tangent vectors. This is the $(n - m)$-dimensional space spanned by $\{\bar{\gamma}^{m+1}, \bar{\gamma}^{m+2}, \ldots, \bar{\gamma}^n\}$.

In a similar fashion, we have a $\binom{m}{p}$-dimensional space of p-vectors which can be identified with each point on the m-dimensional surface. This is the subspace of p-vectors spanned by Clifford numbers of the form $\gamma_{v_1} \wedge \gamma_{v_2} \wedge \ldots \wedge \gamma_{v_p}$ or $\gamma^{v_1} \wedge \gamma^{v_2} \wedge \ldots \wedge \gamma^{v_p}$ where $1 \leqslant v_1 < v_2 < \ldots < v_p \leqslant m$.

(1) Use Eq. (3.38) or (3.39) to show that the orthogonal complement of p-vectors mentioned above is spanned by p-vectors which have the form

$$\gamma_{v_1} \wedge \gamma_{v_2} \wedge \ldots \wedge \gamma_{v_k} \wedge \bar{\gamma}^{v_{k+1}} \wedge \bar{\gamma}^{v_{k+2}} \wedge \ldots \wedge \bar{\gamma}^{v_p}$$

where $0 \leqslant k < p$, $1 \leqslant v_1 < v_2 < \ldots < v_k \leqslant m$, and $m + 1 \leqslant v_{k+1} < v_{k+2} < \ldots < v_p \leqslant n$.

(2) Suppose $\mathscr{F} = F^{v_1 v_2 \cdots v_p}\gamma_{v_1} \wedge \gamma_{v_2} \wedge \ldots \wedge \gamma_{v_p}$. Compute $\partial\mathscr{F}/\partial u^\alpha$ and then use Eq. (4.39) and result (1) to decompose $\partial\mathscr{F}/\partial u^\alpha$ into two p-vectors, one of which is in the space spanned by

$$\{\gamma_{v_1} \wedge \gamma_{v_2} \wedge \ldots \wedge \gamma_{v_p}\} \quad \text{where} \quad 1 \leqslant v_1 < v_2 < \ldots < v_p \leqslant m$$

and another which is in the orthogonal complement.

(3) Use Eqs. (4.41) and (4.46) and result (2) to show $\nabla_\alpha \mathscr{F}$ is the projection of $\partial\mathscr{F}/\partial u^\alpha$ onto the subspace of p-vectors spanned by Clifford numbers of the form $\gamma_{v_1} \wedge \gamma_{v_2} \wedge \ldots \wedge \gamma_{v_p}$ where $1 \leqslant v_1 < v_2 < \ldots < v_p \leqslant m$.

Problem 4.5. Show that if the determinant of $(g_{\alpha\beta})$ is nonzero, then there is no vector (other than $\mathbf{0}$) in the space spanned by the m γ_v's which is orthogonal to that space. Hint: suppose there exists such a vector $\boldsymbol{v}$. Then $\boldsymbol{v} = A^\alpha\gamma_\alpha$ and $0 = \langle \boldsymbol{v}, \gamma_\beta \rangle = A^\alpha\langle \gamma_\alpha, \gamma_\beta \rangle = A^\alpha g_{\alpha\beta}$.

Problem 4.6. Use Eqs. (4.34), (4.35), (4.36), (4.37), and (4.48) to show that for the surface of the sphere $R_{\theta\phi\theta\phi} = -r^2 \sin^2\theta$, and $R_{\theta\phi}{}^{\theta\phi} = -1/r^2$.

Problem 4.7. Use Eq. (4.48) and Eq. (4.41) to show that

$$R_{\alpha\beta\nu}{}^{\eta} = \frac{\partial}{\partial u^\alpha}\Gamma_{\beta\nu}{}^{\eta} - \frac{\partial}{\partial u^\beta}\Gamma_{\alpha\nu}{}^{\eta} + \Gamma_{\beta\nu}{}^{\lambda}\Gamma_{\alpha\lambda}{}^{\eta} - \Gamma_{\alpha\nu}{}^{\lambda}\Gamma_{\beta\lambda}{}^{\eta}.$$

Problem 4.8. Since $\nabla_\alpha \gamma^\beta$ is a tangent vector, it can be expanded as a linear combination of γ^ν's, that is, $\nabla_\alpha \gamma^\beta = C_{\alpha\nu}{}^{\beta}\gamma^\nu$. Show that $C_{\alpha\nu}{}^{\beta} = -\Gamma_{\alpha\nu}{}^{\beta}$. Hint: use the relation that $\delta_\nu^\beta = \langle \gamma_\nu, \gamma^\beta \rangle$ and apply the operator ∇_α to both sides of that equation.

4.3 Parallel Transport on an *m*-Dimensional Surface Embedded in an *n*-Dimensional Flat Space

Having defined the operator ∇_α, it now becomes possible to define the notion of parallel transport of a vector along a curve in an m-dimensional surface embedded in an n-dimensional flat space.

A point on such a curve may be represented by the position vector

$$\boldsymbol{x}(s) = \sum_{i=1}^{n} x^i(u^1(s), u^2(s), \ldots, u^n(s))\hat{\gamma}_\iota. \tag{4.49}$$

To compute the vector tangent to the curve in the direction of increasing s, we merely compute the derivative:

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}s} = \sum_{\alpha=1}^{m}\sum_{\iota=1}^{n} \frac{\partial x^i}{\partial u^\alpha}\frac{\mathrm{d}u^\alpha}{\mathrm{d}s}\hat{\gamma}_\iota = \frac{\mathrm{d}u^\alpha}{\mathrm{d}s}\gamma_\alpha \tag{4.50}$$

where

$$\gamma_\alpha = \frac{\partial x^\iota}{\partial u^\alpha}\hat{\gamma}_\iota.$$

An observer in the large n-dimensional space who wished to compute the derivative of a Clifford number $\mathscr{F}$ with respect to s would simply use the formula

$$\frac{\mathrm{d}\mathscr{F}}{\mathrm{d}s} = \frac{\partial\mathscr{F}}{\partial u^\alpha}\frac{\mathrm{d}u^\alpha}{\mathrm{d}s}. \tag{4.51}$$

However, an observer constrained to take all measurements on the m-dimensional surface would detect only components of $\mathrm{d}\mathscr{F}/\mathrm{d}s$ projected onto the space of Clifford numbers identified with the tangential plane. Thus

such an observer would write

$$\nabla_s \mathscr{F} = \nabla_\alpha \mathscr{F}(\mathrm{d}u^\alpha/\mathrm{d}s).$$

With this definition of ∇_s, we have

$$\nabla_s \gamma_\beta = (\nabla_\alpha \gamma_\beta)(\mathrm{d}u^\alpha/\mathrm{d}s) = \Gamma_{\alpha\beta}{}^\gamma \gamma_\gamma(\mathrm{d}u^\alpha/\mathrm{d}s). \tag{4.52}$$

An index free Clifford number $\mathscr{F}$ which is *parallel transported* along a curve $\boldsymbol{x}(s)$ (not necessarily a geodesic) is one for which

$$\nabla_s \mathscr{F} = \mathbf{0}. \tag{4.53}$$

We can now give a more sophisticated definition of a geodesic than that given in Section 4.1. A *geodesic* is a curve $\boldsymbol{x}(s)$ which can be parameterized in terms of a variable s in such a way that the tangent vector $\mathrm{d}\boldsymbol{x}/\mathrm{d}s$ is parallel transported along the curve; that is

$$\nabla_s \left(\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}s}\right) = \nabla_s^2 \boldsymbol{x}(s) = \mathbf{0}. \tag{4.54}$$

It should be emphasized that the parameterization of a geodesic may be such that Eq. (4.54) does not hold. This is most easily visualized if s is thought of as a time variable. Then $\boldsymbol{x}(s)$ represents the position of a point which moves along our surface as time advances. The tangent vector $\mathrm{d}\boldsymbol{x}/\mathrm{d}s$ then represents the velocity vector. Of course we can adjust the parameterization so that the point moves with a changing speed. Clearly if the magnitude of the velocity is changing then the acceleration $\nabla_s^2 \boldsymbol{x}(s)$ measured even by an observer constrained to the surface is not zero. A geodesic on a surface may be considered to be a path such that a point moving along it with constant speed will experience no acceleration or change in direction which is detectable to an observer who is constrained to take all measurements on that surface.

To obtain the equations for the coordinates of a geodesic, one can use Eqs. (4.50) and (4.52) to get

$$\begin{aligned}
\nabla_s^2 \boldsymbol{x}(s) &= \nabla_s\left(\gamma_\beta \frac{\mathrm{d}u^\beta}{\mathrm{d}s}\right) \\
&= (\nabla_\alpha \gamma_\beta)\frac{\mathrm{d}u^\alpha}{\mathrm{d}s}\frac{\mathrm{d}u^\beta}{\mathrm{d}s} + \gamma_\beta \frac{\mathrm{d}^2 u^\beta}{\mathrm{d}s^2} \\
&= \gamma_\nu\left(\Gamma_{\alpha\beta}{}^\nu \frac{\mathrm{d}u^\alpha}{\mathrm{d}s}\frac{\mathrm{d}u^\beta}{\mathrm{d}s} + \frac{\mathrm{d}^2 u^\nu}{\mathrm{d}s^2}\right).
\end{aligned}$$

Therefore the equations for the coordinates of a geodesic may be written as

$$\frac{\mathrm{d}^2 u^\nu}{\mathrm{d}s^2} + \Gamma_{\alpha\beta}{}^\nu \frac{\mathrm{d}u^\alpha}{\mathrm{d}s}\frac{\mathrm{d}u^\beta}{\mathrm{d}s} = 0. \tag{4.55}$$

Of course if the u^α's are Euclidean coordinates in a Euclidean plane, then the Christoffel symbols are zero and the equations ar easily solved. One should not be too surprised to observe that in this situation the general solution for a geodesic is an arbitrary straight line in the m-dimensional Euclidean plane; that is

$$u^\alpha = c^\alpha s + u_0^\alpha \quad \text{for } \alpha = 1, 2, \ldots, m,$$

where the c^α's and u_0^α's are arbitrary constants.

Problem 4.9. From Eqs. (4.34)–(4.37), one can read off the values for the Christoffel symbols for the surface of a sphere. Use these values and Eq. (4.53) to treat the problem of parallel transporting a vector around a four-sided figure on a sphere formed by two parallels and two meridians. That is, first parallel transport the vector $A^\theta\gamma_0 + A^\phi\gamma_\phi$ "south" along the path $\theta = s + \theta_0$, $\phi = \phi_0$ from $s = 0$ to $s = \theta_1 - \theta_0$. Then parallel transport the vector "east" along the path $\theta = \theta_1$, $\phi = s + \phi_0$ from $s = 0$ to $s = \phi_1 - \phi_0$. Then parallel transport the vector "north" along the path $\theta = -s + \theta_1$, $\phi = \phi_1$ from $s = 0$ to $s = \theta_1 - \theta_0$. Finally parallel transport the vector back to its original position by moving it "west" along the path $\theta = \theta_0$ and $\phi = -s + \phi_1$ from $s = 0$ to $s = \phi_1 - \phi_0$.

Several checkpoints for the computation are as follows. For all path segments, one should get

$$\frac{\partial A^\theta}{\partial\theta}\frac{\mathrm{d}\theta}{\mathrm{d}s} + \frac{\partial A^\theta}{\partial\phi}\frac{\mathrm{d}\phi}{\mathrm{d}s} - A^\phi \sin\theta\cos\theta\frac{\mathrm{d}\phi}{\mathrm{d}s} = 0$$

and

$$\frac{\partial A^\phi}{\partial\theta}\frac{\mathrm{d}\theta}{\mathrm{d}s} + A^\phi\frac{\cos\theta}{\sin\theta}\frac{\mathrm{d}\theta}{\mathrm{d}s} + A^\theta\frac{\cos\theta}{\sin\theta}\frac{\mathrm{d}\phi}{\mathrm{d}s} = 0.$$

Integrating these equations along the first leg of the rectangular loop, one should get

$$A^\theta = A_0^\theta \qquad \text{and} \qquad A^\phi \sin\theta = A_0^\phi \sin\theta_0$$

where (A_0^θ, A_0^ϕ) are the components of the vector in the northwest corner of the loop.

Integrating along the second segment, one should get

$$A^\theta = A_1^\theta \cos[(\phi - \phi_0)\cos\theta_1] + (A_1^\phi \sin\theta_1)\sin[(\phi - \phi_1)\cos\theta_1]$$

and

$$A^\phi \sin\theta_1 = (A_1^\phi \sin\theta_1)\cos[(\phi - \phi_0)\cos\theta_1] - A_1^\theta \sin[(\phi - \phi_1)\cos\theta_1]$$

where (A_1^θ, A_1^ϕ) are the components of the vector in the southwest corner of the loop. Note: the parallel transport along the curve of constant latitude results in a clockwise rotation with respect to the curve of constant latitude. This is the same direction of rotation that would result if you approximated the parallel by a sequence of geodesics.

When you have completed the parallel transport of the vector about the rectangular loop, you should be able to show that the resulting angle of rotation is $(\phi_1 - \phi_0)(\cos\theta_0 - \cos\theta_1)$. It is an easy matter to compute the area of the rectangular loop so that one can see that the result for this problem is consistent with Eq. (4.3).

5

THE USE OF FOCK–IVANENKO 2-VECTORS TO OBTAIN THE SCHWARZSCHILD METRIC

5.1 The Operator ∇_α and Dirac Matrices in Curved Spaces

In Chapter 10, it will be shown how to construct a system of orthonormal Dirac matrices for an arbitrary signature matrix. That is, given any signature matrix n_{jk}, it is possible to construct a set of Dirac matrices such that

$$\hat{\gamma}_j\hat{\gamma}_k + \hat{\gamma}_j\hat{\gamma}_k = 2n_{jk}I. \tag{5.1}$$

Using this result, it is not too difficult to construct a set of coordinate Dirac matrices for a more general metric tensor.

Suppose we denote the matrix corresponding to the matrix tensor $g_{\alpha\beta}$ by G. That is

$$G = \begin{bmatrix} g_{11} & g_{12} & \cdots & g_{1n} \\ g_{21} & g_{22} & \cdots & g_{2n} \\ \vdots & & & \vdots \\ g_{n1} & g_{n2} & \cdots & g_{nn} \end{bmatrix}.$$

An arbitrary metric tensor is real and symmetric, that is $g_{\alpha\beta} = g_{\beta\alpha}$. From matrix theory it is well known that a real symmetric matrix can be diagonalized by a real orthogonal matrix. Thus we have

$$G = O^{\mathrm{T}}DO \tag{5.2}$$

where O is an orthogonal matrix; O^{T} is the transpose of O which is identical to O^{-1}; and D is a diagonal matrix. Furthermore, the diagonal elements of D are real and may be ordered so that all the positive components are

identified with the first r indices. That is

$$D = \begin{bmatrix} (\alpha_1)^2 & & & & & & \\ & (\alpha_2)^2 & & & & & \\ & & \ddots & & & & \\ & & & (\alpha_r)^2 & & & \\ & & & & -(\alpha_{r+1})^2 & & \\ & & & & & \ddots & \\ & & & & & & -(\alpha_{r+s})^2 \end{bmatrix}$$

where $r + s = n$. Clearly D can be decomposed further into a product of the form

$$D = AnA \tag{5.3}$$

where n is the signature matrix and

$$A = \begin{bmatrix} \alpha_1 & 0 & \dots & 0 \\ 0 & \alpha_2 & \dots & 0 \\ & & \ddots & \\ 0 & & \dots & \alpha_n \end{bmatrix}.$$

The signature matrix n is of course uniquely determined by the metric tensor.

From Eqs. (5.2) and (5.3), we have

$$G = O^{\mathrm{T}}A^{\mathrm{T}}nAO = W^{\mathrm{T}}nW \quad \text{where } W = AO.$$

Thus

$$g_{\alpha\beta} = W^j_\alpha W^k_\beta n_{jk}. \tag{5.4}$$

If we now define $\gamma_\alpha = W^j_\alpha \hat{\gamma}_j$ then

$$\begin{aligned} \gamma_\alpha\gamma_\beta + \gamma_\beta\gamma_\alpha &= W^j_\alpha W^k_\beta(\hat{\gamma}_j\hat{\gamma}_k + \hat{\gamma}_k\hat{\gamma}_j) \\ &= 2W^j_\alpha W^k_\beta n_{jk} I \\ &= 2g_{\alpha\beta} I. \end{aligned} \tag{5.5}$$

To obtain the contravariant Dirac matrices, we merely write

$$\gamma^\alpha = g^{\alpha\nu}\gamma_\nu. \tag{5.6}$$

In Chapter 4, we discussed m-dimensional surfaces embedded in an n-dimensional Euclidean or pseudo-Euclidean space. In that context $\nabla_\alpha \gamma_\beta$ was defined to be the projection of $\partial\gamma_\beta/\partial u^\alpha$ onto the m-dimensional tangent space spanned by $\gamma_1, \gamma_2, \ldots, \gamma_m$ at a given point on the m-dimensional surface under study. In that situation, it was noted that

$$\nabla_\alpha \gamma_\beta = \nabla_\beta \gamma_\alpha. \tag{5.7}$$

In a more general context, an m-dimensional curved space may or may not be embedded in a higher dimensional flat space. In this more general context, a differential operator that satisfies Eq. (5.7) is said to be *torsion-free*. The operator ∇_α is uniquely defined by Eq. (5.7) and four other conditions which are even more compelling. The five conditions are:

(1) ∇_α acting on a scalar or the tensor component of any p-vector coincides with the derivative $\partial/\partial u^\alpha$;
(2) $\nabla_\alpha \gamma_\beta = \nabla_\beta \gamma_\alpha$ (∇_α is torsion-free);
(2) $\nabla_\alpha \gamma_\beta$ is a linear combination of the γ_ν's, that is $\nabla_\alpha \gamma_\beta = \Gamma_{\alpha\beta}{}^\nu \gamma_\nu$;
(4) if $\mathscr{A}$ and $\mathscr{B}$ are any Clifford numbers with differentiable components, then

$$\nabla_\alpha(\mathscr{A} + \mathscr{B}) = \nabla_\alpha \mathscr{A} + \nabla_\alpha \mathscr{B};$$

(5) ∇_α satisfies the Leibniz property, that is

$$\nabla_\alpha(\mathscr{A}\mathscr{B}) = (\nabla_\alpha \mathscr{A})\mathscr{B} + \mathscr{A}(\nabla_\alpha \mathscr{B}).$$

To see that these five conditions do indeed define ∇_α, we need only see how these five conditions determine a formula for the Christoffel symbols. Since $2g_{\alpha\beta}I = \gamma_\alpha\gamma_\beta + \gamma_\beta\gamma_\alpha$, we have

$$\nabla_\nu(2g_{\alpha\beta}I) = \nabla_\nu(\gamma_\alpha\gamma_\beta + \gamma_\beta\gamma_\alpha). \tag{5.8}$$

Using condition (1) on the left-hand side of Eq. (5.8) and conditions (4) and (5) on the right-hand side, we get

$$2\frac{\partial g_{\alpha\beta}}{\partial u^\nu} I = (\nabla_\nu \gamma_\alpha)\gamma_\beta + \gamma_\beta(\nabla_\nu \gamma_\alpha) + \gamma_\alpha(\nabla_\nu \gamma_\beta) + (\nabla_\nu \gamma_\beta)\gamma_\alpha.$$

Now applying condition (3) on the right-hand side, one has

$$2\frac{\partial g_{\alpha\beta}}{\partial u^\nu} I = \Gamma_{\nu\alpha}{}^\eta(\gamma_\eta\gamma_\beta + \gamma_\beta\gamma_\eta) + \Gamma_{\nu\beta}{}^\eta(\gamma_\alpha\gamma_\eta + \gamma_\eta\gamma_\alpha),$$

or

$$\frac{\partial g_{\alpha\beta}}{\partial u^{\nu}} = \Gamma_{\nu\alpha}{}^{\eta} g_{\eta\beta} + \Gamma_{\nu\beta}{}^{\eta} g_{\alpha\eta}. \tag{5.9}$$

By cyclic permutation of the unsummed indices, we obtain two other equations:

$$\frac{\partial g_{\beta\nu}}{\partial u^{\alpha}} = \Gamma_{\alpha\beta}{}^{\eta} g_{\eta\nu} + \Gamma_{\alpha\nu}{}^{\eta} g_{\beta\eta} \tag{5.10}$$

and

$$-\frac{\partial g_{\nu\alpha}}{\partial u^{\beta}} = -\Gamma_{\beta\nu}{}^{\eta} g_{\eta\alpha} - \Gamma_{\beta\alpha}{}^{\eta} g_{\nu\eta}. \tag{5.11}$$

From the torsion-free condition, $\Gamma_{\nu\beta}{}^{\eta} = \Gamma_{\beta\nu}{}^{\eta}$. Thus if we add Eqs. (5.9), (5.10), and (5.11), we get

$$2\Gamma_{\nu\alpha}{}^{\eta} g_{\eta\beta} = \frac{\partial g_{\alpha\beta}}{\partial u^{\nu}} + \frac{\partial g_{\beta\nu}}{\partial u^{\alpha}} - \frac{\partial g_{\nu\alpha}}{\partial u^{\beta}}.$$

Multiplying both sides of this equation by $g^{\lambda\beta}/2$ and noting that $g_{\eta\beta} g^{\lambda\beta} = \delta^{\lambda}_{\eta}$, one immediately obtains the equation

$$\Gamma_{\nu\alpha}{}^{\lambda} = \frac{g^{\lambda\beta}}{2}\left(\frac{\partial g_{\alpha\beta}}{\partial u^{\nu}} + \frac{\partial g_{\beta\nu}}{\partial u^{\alpha}} - \frac{\partial g_{\nu\alpha}}{\partial u^{\beta}}\right). \tag{5.12}$$

The operator ∇_α may be applied to any Clifford number with differentiable components including contravariant Dirac matrices. Since the contravariant Dirac matrix γ^β is a linear combination of covariant γ_ν's, it is clear that $\nabla_\alpha \gamma^\beta$ is a linear combination of γ_ν's and is therefore also a linear combination of γ^η's. Thus $\nabla_\alpha \gamma^\beta$ can be written in the form $A_{\alpha\eta}{}^{\beta}\gamma^\eta$. It is not difficult to show that $A_{\alpha\eta}{}^{\beta} = -\Gamma_{\alpha\eta}{}^{\beta}$. To prove this identity, we observe that

$$\begin{aligned}
\mathbf{0} &= \nabla_\alpha(2\delta^{\beta}_{\eta} I) \\
&= \nabla_\alpha(\gamma^\beta\gamma_\eta + \gamma_\eta\gamma^\beta) \\
&= (\nabla_\alpha\gamma^\beta)\gamma_\eta + \gamma_\eta(\nabla_\alpha\gamma^\beta) + \gamma^\beta(\nabla_\alpha\gamma_\eta) + (\nabla_\alpha\gamma_\eta)\gamma^\beta \\
&= A_{\alpha\nu}{}^{\beta}(\gamma^\nu\gamma_\eta + \gamma_\eta\gamma^\nu) + \Gamma_{\alpha\eta}{}^{\nu}(\gamma^\beta\gamma_\nu + \gamma_\nu\gamma^\beta) \\
&= 2A_{\alpha\nu}{}^{\beta}(\delta^{\nu}_{\eta} I) + 2\Gamma_{\alpha\eta}{}^{\nu}(\delta^{\beta}_{\nu} I).
\end{aligned}$$

Therefore $0 = A_{\alpha\eta}{}^{\beta} + \Gamma_{\alpha\eta}{}^{\beta}$ and

$$\nabla_\alpha\gamma^\beta = -\Gamma_{\alpha\eta}{}^{\beta}\gamma^\eta. \tag{5.13}$$

It is important to observe that the Christoffel symbols do not transform

as tensors. From Eq. (5.8), we have

$$\frac{\partial}{\partial x^\nu}(2g_{\alpha\beta}I) = \nabla_\nu(\gamma_\alpha\gamma_\beta + \gamma_\beta\gamma_\alpha).$$

It was shown that this equation essentially defines ∇_ν. Therefore under a change of coordinates, ∇_ν behaves in the same manner as $\partial/\partial u^\nu$. That is

$$\bar{\nabla}_\alpha = \frac{\partial u^\eta}{\partial \bar{u}^\alpha}\nabla_\eta. \tag{5.14}$$

Thus

$$\bar{\nabla}_\alpha \bar{\gamma}^\beta = \left(\frac{\partial u^\eta}{\partial \bar{u}^\alpha}\nabla_\eta\right)\left(\gamma^\nu \frac{\partial \bar{u}^\beta}{\partial u^\nu}\right)$$

and

$$-\bar{\Gamma}_{\alpha\lambda}{}^\beta \bar{\gamma}^\lambda = -\frac{\partial u^\eta}{\partial \bar{u}^\alpha}\frac{\partial \bar{u}^\beta}{\partial u^\nu}\Gamma_{\eta\mu}{}^\nu \gamma^\mu + \frac{\partial u^\eta}{\partial \bar{u}^\alpha}\frac{\partial^2 \bar{u}^\beta}{\partial u^\eta\, \partial u^\nu}\gamma^\nu.$$

It then follows that

$$\bar{\Gamma}_{\alpha\lambda}{}^\beta \bar{\gamma}^\lambda = \frac{\partial u^\eta}{\partial \bar{u}^\alpha}\frac{\partial \bar{u}^\beta}{\partial u^\nu}\frac{\partial u^\mu}{\partial \bar{u}^\lambda}\Gamma_{\eta\mu}{}^\nu \bar{\gamma}^\lambda - \frac{\partial u^\eta}{\partial \bar{u}^\alpha}\frac{\partial u^\nu}{\partial \bar{u}^\lambda}\frac{\partial^2 \bar{u}^\beta}{\partial u^\eta\, \partial u^\nu}\bar{\gamma}^\lambda$$

and finally

$$\bar{\Gamma}_{\alpha\lambda}{}^\beta = \frac{\partial u^\eta}{\partial \bar{u}^\alpha}\frac{\partial \bar{u}^\beta}{\partial u^\nu}\frac{\partial u^\mu}{\partial \bar{u}^\lambda}\Gamma_{\eta\mu}{}^\nu - \frac{\partial u^\eta}{\partial \bar{u}^\alpha}\frac{\partial u^\nu}{\partial \bar{u}^\lambda}\frac{\partial^2 \bar{u}^\beta}{\partial u^\eta\, \partial u^\nu}. \tag{5.15}$$

Equation (5.15) can also be proven with more grief directly from Eq. (5.12).

Although Christoffel symbols do not transform as tensors, we see from Eq. (5.14) that $\nabla_\alpha \mathscr{A}$ does transform as a tensor. It is understood here that $\mathscr{A}$ is a Clifford number that is *index free*, that is $\mathscr{A}$ has no unsummed indices. For example, suppose $\mathscr{A} = A_{\beta\nu}{}^\eta \gamma^\beta\gamma^\nu\gamma_\eta$. Then

$$\begin{aligned}\nabla_\alpha \mathscr{A} &= (\partial A_{\beta\nu}{}^\eta/\partial u^\alpha)\gamma^\beta\gamma^\nu\gamma_\eta - A_{\beta\nu}{}^\eta(\Gamma_{\alpha\lambda}{}^\beta\gamma^\lambda)\gamma^\nu\gamma_\eta \\ &\quad - A_{\beta\nu}{}^\eta\gamma^\beta(\Gamma_{\alpha\lambda}{}^\nu\gamma^\lambda)\gamma_\eta + A_{\beta\nu}{}^\eta\gamma^\beta\gamma^\nu(\Gamma_{\alpha\eta}{}^\lambda\gamma_\lambda).\end{aligned} \tag{5.16}$$

Introducing the *comma notation*, we can write

$$\frac{\partial A_{\beta\nu}{}^\eta}{\partial u^\alpha} = A_{\beta\nu}{}^\eta{}_{,\alpha}. \tag{5.17}$$

Relabeling some of the dummy indices in Eq. (5.16), we now have

$$\nabla_\alpha \mathscr{A} = (A_{\beta\nu}{}^{\eta}{}_{,\alpha} - A_{\lambda\nu}{}^{\eta}\Gamma_{\alpha\beta}{}^{\lambda} - A_{\beta\lambda}{}^{\eta}\Gamma_{\alpha\nu}{}^{\lambda} + A_{\beta\nu}{}^{\lambda}\Gamma_{\alpha\lambda}{}^{\eta})\gamma^\beta\gamma^\nu\gamma_\eta \tag{5.18}$$

This motivates the introduction of the *semicolon notation.* In our example

$$A_{\beta\nu}{}^{\eta}{}_{;\alpha} = A_{\beta\nu}{}^{\eta}{}_{,\alpha} - A_{\lambda\nu}{}^{\eta}\Gamma_{\alpha\beta}{}^{\lambda} - A_{\beta\lambda}{}^{\eta}\Gamma_{\alpha\nu}{}^{\lambda} + A_{\beta\nu}{}^{\lambda}\Gamma_{\alpha\lambda}{}^{\eta}. \tag{5.19}$$

With this notation Eq. (5.18) becomes

$$\nabla_\alpha \mathscr{A} = A_{\beta\nu}{}^{\eta}{}_{;\alpha}\gamma^\beta\gamma^\nu\gamma_\eta. \tag{5.20}$$

Both of the entities $\nabla_\alpha \mathscr{A}$ and $\gamma^\beta\gamma^\nu\gamma_\eta$ transform as tensors and therefore $A_{\beta\nu}{}^{\eta}{}_{;\alpha}$ also transforms as a tensor. That is

$$\bar{A}_{\mu\lambda}{}^{\delta}{}_{;\rho} = \frac{\partial u^\beta}{\partial \bar{u}^\mu}\frac{\partial u^\nu}{\partial \bar{u}^\lambda}\frac{\partial \bar{u}^\delta}{\partial u^\eta}\frac{\partial u^\alpha}{\partial \bar{u}^\rho} A_{\beta\nu}{}^{\eta}{}_{;\alpha} \tag{5.21}$$

For many years $A_{\beta\nu}{}^{\eta}{}_{;\alpha}$ was said to be the *covariant derivative* of the tensor $A_{\beta\nu}{}^{\eta}$. In recent years some people have gone to great lengths to formulate matters in a way which would avoid reference to anything which could be interpreted as a specific coordinate system. To pursue this end, a slightly modified terminology and notation is used. In *Spinors and Space-Time* by Penrose and Rindler (1984) and *General Relativity* by Wald (1984), an "abstract index" is used. In our example the Clifford number $\mathscr{A}$ would be labeled $A_{bc}{}^{d}$. The indices b, c, and d merely indicate the type of tensor components in the index free Clifford number and do not refer to any coordinate system. In the same spirit, $\gamma^\alpha\nabla_\alpha$ is labeled by ∇_a and is referred to as the "covariant derivative". In this context $A_{\beta\nu}{}^{\rho}{}_{;\alpha}$ is said to be a "component" of the "tensor" $\nabla_a A_{bc}{}^{d}$.

For our purposes, it will be more useful to use the traditional terminology. To extend the notion of covariant derivative to coordinate Dirac matrices, we require that the covariant derivative have the Leibniz property and we also require that if $\mathscr{A}$ is any index free Clifford number then

$$\nabla_\alpha \mathscr{A} = \mathscr{A}_{;\alpha}. \tag{5.22}$$

This means that if $\mathscr{A} = A^\beta\gamma_\beta$, then $\nabla_\alpha(A^\beta\gamma_\beta) = (A^\beta\gamma_\beta)_{,\alpha}$ or

$$A^\beta{}_{;\alpha}\gamma_\beta + A^\beta\Gamma_{\alpha\beta}{}^{\eta}\gamma_\eta = A^\beta{}_{;\alpha}\gamma_\beta + A^\beta\gamma_{\beta;\alpha}. \tag{5.23}$$

Relabeling some dummy indices on the left-hand side of Eq. (5.23), we have

$$(A^\beta{}_{,\alpha} + A^\eta\Gamma_{\alpha\eta}{}^{\beta})\gamma_\beta = A^\beta{}_{;\alpha}\gamma_\beta + A^\beta\gamma_{\beta;\alpha}.$$

Since $A^{\beta}{}_{;\alpha} = A^{\beta}{}_{,\alpha} + A^{\eta}\Gamma_{\alpha\eta}{}^{\beta}$ and since the A^{β}'s are arbitrary it follows that

$$\gamma_{\beta;\alpha} = \mathbf{0}. \tag{5.24}$$

In a similar fashion, it can be shown that

$$\gamma^{\beta}{}_{,\alpha} = \mathbf{0}. \tag{5.25}$$

Problem 5.1. Show that $A^{\alpha}{}_{,\beta}$ is not a tensor.

Problem 5.2. Use Eq. (5.24) to show that $g_{\alpha\beta;\nu} = 0$.

Problem 5.3. Suppose $A^{\alpha_1\alpha_2\ldots\alpha_r}{}_{\beta_1\beta_2\ldots\beta_s}$ transforms as a tensor. Show that the *contracted tensor* $A^{\alpha_1\alpha_2\ldots\alpha_{r-1}\lambda}{}_{\beta_1\beta_2\ldots\beta_{s-1}\lambda}$ also transforms as a tensor but as one of lower order.

Problem 5.4. Show that

$$A_{\alpha;\beta}\gamma^{\beta\alpha} = \frac{\partial A_\alpha}{\partial u^\beta}\gamma^{\beta\alpha}.$$

Problem 5.5. Show that the result of Problem 5.4 can be generalized to higher order p-vectors. That is

$$A_{\alpha_1\alpha_2\ldots\alpha_n;\beta}\gamma^{\beta\alpha_1\alpha_2\ldots\alpha_n} = A_{\alpha_1\alpha_2\ldots\alpha_n,\beta}\gamma^{\beta\alpha_1\alpha_2\ldots\alpha_n}.$$

5.2 Connection Coefficients and Fock–Ivanenko 2-Vectors

So far, we have focused mostly on coordinate systems of Dirac matrices. However, for many computations, it is useful to use noncoordinate systems. For systems which are possibly noncoordinate, we will use Latin indices and we will continue to reserve Greek indices for systems which are specifically coordinate. For a possibly noncoordinate system, one has

$$\nabla_j\gamma_k = W_j^\alpha\nabla_\alpha(W_k^\beta\gamma_\beta) = W_j^\alpha W_k^\beta\Gamma_{\alpha\beta}{}^{\eta}\gamma_\eta + W_j^\alpha W_{k,\alpha}^\beta\gamma_\beta. \tag{5.26}$$

We define $\Gamma_{jk}{}^{m}$ by the relation

$$\nabla_j\gamma_k = \Gamma_{jk}{}^{m}\gamma_m. \tag{5.27}$$

These generalized Christoffel symbols are called *connection coefficients.* For a coordinate system that is torsion-free, $\Gamma_{\alpha\beta}{}^{\eta} = \Gamma_{\beta\alpha}{}^{\eta}$. However, for a noncoordinate system, it is generally not true that $\Gamma_{jk}{}^{m} = \Gamma_{kj}{}^{m}$. Therefore, it is important to pay attention to the order of the lower indices in the connection coefficients. Traditionally those using the comma and semicolon notation

have used the order opposite to that used in Eq. (5.27). However, those using the ∇_k notation find the sequence adopted in this text to be more natural. For this reason, I have followed the convention of Penrose and Rindler (1984).

From Eqs. (5.26) and (5.27), we have

$$\Gamma_{jk}{}^{m}\gamma_m = W_j^\alpha W_k^\beta W_\eta^m \Gamma_{\alpha\beta}{}^{\eta}\gamma_m + W_j^\alpha W_{k,\alpha}^\beta W_\beta^m \gamma_m,$$

and thus

$$\Gamma_{jk}{}^{m} = \tfrac{1}{2} W_j^\alpha W_k^\beta W_\eta^m (g^{\eta\nu}) \left(\frac{\partial g_{\nu\beta}}{\partial u^\alpha} + \frac{\partial g_{\nu\alpha}}{\partial u^\beta} - \frac{\partial g_{\alpha\beta}}{\partial u^\nu} \right) + W_j^\alpha W_\beta^m W_{k,\alpha}^\beta. \tag{5.28}$$

We now observe that

$$W_\eta^m g^{\eta\nu} = W_\eta^m (W_p^\eta W_q^\nu g^{pq}) = \delta_p^m W_q^\nu g^{pq} = W_q^\nu g^{mq}.$$

Thus Eq. (5.28) becomes

$$\Gamma_{jk}{}^{m} = \tfrac{1}{2} W_j^\alpha W_k^\beta W_q^\nu g^{mq} \left(\frac{\partial g_{\nu\beta}}{\partial u^\alpha} + \frac{\partial g_{\nu\alpha}}{\partial u^\beta} - \frac{\partial g_{\alpha\beta}}{\partial u^\nu} \right) + W_j^\alpha W_\beta^m W_{k,\alpha}^\beta \tag{5.29}$$

or

$$\Gamma_{jk}{}^{m} = \tfrac{1}{2} W_j^\alpha W_k^\beta W_q^\nu g^{mq}(A) + W_j^\alpha W_\beta^m W_{k,\alpha}^\beta \tag{5.30}$$

where

$$A = \frac{\partial}{\partial u^\alpha}(W_\nu^r W_\beta^s g_{rs}) + \frac{\partial}{\partial u^\beta}(W_\nu^r W_\alpha^s g_{rs}) - \frac{\partial}{\partial u^\nu}(W_\alpha^r W_\beta^s g_{rs}).$$

To reorganize this formula into a standard form, it is necessary to introduce the notion of *commutator coefficients* $c_{jk}{}^{m}$. These coefficients are defined by the relation

$$(\partial_j \partial_k - \partial_k \partial_j) = c_{jk}{}^{m} \partial_m \tag{5.31}$$

where

$$\partial_j = W_j^\alpha \frac{\partial}{\partial u^\alpha}.$$

From this definition, a small amount of scratch work will show that

$$c_{jk}{}^{m} = W_\alpha^m (\partial_j W_k^\alpha - \partial_k W_j^\alpha). \tag{5.32}$$

After a substantial amount of manipulation Eq. (5.30) becomes

$$\Gamma_{jk}{}^{m} = \tfrac{1}{2} g^{mq}(\partial_j g_{kq} + \partial_k g_{jq} - \partial_q g_{jk}) + \tfrac{1}{2} g^{mq}(c_{qkj} + c_{jkq} - c_{jqk}) \tag{5.33}$$

where

$$c_{jkp} = g_{pm}c_{jk}{}^{m}. \tag{5.34}$$

We now define Γ_{jkp} by the equation

$$\Gamma_{jkp} = g_{pm}\Gamma_{jk}{}^{m}. \tag{5.35}$$

It should be noted that this is not standard notation. Usually when the upper index of a Christoffel symbol or connection symbol is lowered, it is lowered into the first position. That is $\Gamma_{jkp} = g_{jp}\Gamma^{q}{}_{kp}$. I have chosen Eq. (5.35) to define Γ_{jkp} because it is more compatible with the ∇_j notation. With this convention, it follows from Eq. (5.33) that

$$\Gamma_{jkp} = \tfrac{1}{2}(\partial_j g_{kp} + \partial_k g_{jp} - \partial_p g_{jk}) + \tfrac{1}{2}(c_{pkj} + c_{jkp} - c_{jpk}). \tag{5.36}$$

Suppose an orthonormal system of Dirac matrices is chosen. In that circumstance, the Dirac matrices can be regarded as constant with respect to ordinary differentiation, that is

$$\frac{\partial}{\partial u^{\alpha}}\hat{\gamma}_k = 0.$$

Then

$$\partial_j\hat{\gamma}_k = W_j^{\alpha}\frac{\partial}{\partial u^{\alpha}}\hat{\gamma}_k = 0.$$

In this situation, the first three terms in Eq. (5.36) are zero. In that case

$$\Gamma_{jkp} = \tfrac{1}{2}(c_{pkj} + c_{jkp} - c_{jpk}). \tag{5.37}$$

In this context, the connection coefficients are known as *Ricci rotation coefficients.* From Eqs. (5.32), (5.34), and (5.37), it follows that

$$\Gamma_{jkp} = -\Gamma_{jpk}. \tag{5.38}$$

This symmetry can result in substantial savings in computation. In an n-dimensional coordinate system where $\Gamma_{\alpha\beta}{}^{\eta} = \Gamma_{\beta\alpha}{}^{\eta}$ there are as many as $\binom{n}{2}n$ distinct nonzero Christoffel symbols for which $\alpha \neq \beta$ and n^2 more symbols for which $\alpha = \beta$. On the other hand for a constant frame there are only $\binom{n}{2}n$ Ricci rotation coefficients to compute. For the usual 4-dimensional space–time manifold of general relativity, this means computing 24 entities instead of 40. Furthermore, if the coordinate matrix tensor $g_{\alpha\beta}$ is diagonal, even further savings occur. In that case the constant frame can be chosen so that $W_{\alpha}^{a} = 0$ unless $a = \alpha$. From Eqs. (5.32) and (5.34), it is clear

that for this case $c_{jkp} = 0$ unless $p = k$ or $p = j$. From Eq. (5.37), $\Gamma_{jkp} = 0$ unless at least two of the indices are identical. Because of the antisymmetry of the last two indices $\Gamma_{jkp} = 0$ if $k = p$. Thus the only non-zero connection coefficients in this case are of the form Γ_{jjp} or Γ_{jpj}. Since $\Gamma_{jjp} = -\Gamma_{jpj}$ and $p \neq j$, there are only $n(n-1)$ nonzero coefficients to compute. For the 4-dimensional space-time manifold of general relativity, this means a maximum of 12 non-zero connection coefficients when the coordinate metric tensor is diagonal.

To take full advantage of these symmetries, it is useful to apply the notion of *Fock–Ivanenko 2-vectors* (Fock and Ivanenko 1929; Fock 1929). A Fock–Ivanenko 2-vector $\mathbf{\Gamma}_\alpha$ may be defined by the equation

$$\mathbf{\Gamma}_\alpha = \tfrac{1}{4} W^i_\alpha \Gamma_{ijk} \hat{\gamma}^{jk}. \tag{5.39}$$

What makes these 2-forms useful is the relation that if $\hat{\gamma}_k$ is a member of a constant system of Dirac matrices for which $\hat{\gamma}_{k,\alpha} = 0$, then

$$\nabla_\alpha \hat{\gamma}_k = -\mathbf{\Gamma}_\alpha \hat{\gamma}_k + \hat{\gamma}_k \mathbf{\Gamma}_\alpha. \tag{5.40}$$

To verify Eq. (5.40), it is useful to note that

$$\gamma^p \gamma^q \gamma_k = \gamma^p(\gamma^q \gamma_k + \gamma_k \gamma^q) - (\gamma^p \gamma_k + \gamma_k \gamma^p)\gamma^q + \gamma_k \gamma^p \gamma^q$$

or

$$\gamma^{pq}\gamma_k - \gamma_k \gamma^{pq} = 2\gamma^p\, \delta^q_k - 2\gamma^q\, \delta^p_k. \tag{5.41}$$

Thus

$$\begin{aligned}
-\mathbf{\Gamma}_\alpha \hat{\gamma}_k + \hat{\gamma}_k \mathbf{\Gamma}_\alpha &= -\tfrac{1}{4} W^i_\alpha \Gamma_{ipq}(\hat{\gamma}^{pq}\hat{\gamma}_k - \hat{\gamma}_k \hat{\gamma}^{pq}) \\
&= -\tfrac{1}{2} W^i_\alpha \Gamma_{ipq}(\hat{\gamma}^p \delta^q_k - \hat{\gamma}^q \delta^p_k) \\
&= \tfrac{1}{2} W^i_\alpha(-\Gamma_{ipk}\hat{\gamma}^p + \Gamma_{ikq}\hat{\gamma}^q) \\
&= W^i_\alpha \Gamma_{ikq}\hat{\gamma}^q \\
&= W^i_\alpha \Gamma_{ik}{}^p \hat{\gamma}_p \\
&= W^i_\alpha \nabla_i \hat{\gamma}_k \\
&= \nabla_\alpha \hat{\gamma}_k.
\end{aligned}$$

As will be shown below, Eq. (5.40) can be generalized in a nice way to any Clifford number. Meanwhile, we wish to establish a simple formula for Γ_α. From Eq. (5.29), one sees that

$$\Gamma_{ijp} = \tfrac{1}{2} W^\mu_i W^\beta_j W^\nu_p \left(\frac{\partial g_{\nu\beta}}{\partial u^\mu} + \frac{\partial g_{\nu\mu}}{\partial u^\beta} - \frac{\partial g_{\mu\beta}}{\partial u^\nu} \right) + n_{mp} W^\mu_i W^m_\beta \frac{\partial W^\beta_j}{\partial u^\mu}.$$

It then follows that

$$\begin{aligned}\mathbf{\Gamma}_\alpha &= \tfrac{1}{4}W^i_\alpha \Gamma_{ijq}\hat{\gamma}^{jq} \\ &= \tfrac{1}{8}(\delta^\mu_\alpha)(W^\beta_j W^\nu_q \hat{\gamma}^{jq})\left(\frac{\partial g_{\nu\beta}}{\partial u^\mu} + \frac{\partial g_{\nu\mu}}{\partial u^\beta} - \frac{\partial g_{\mu\beta}}{\partial u^\nu}\right) \\ &\quad + \tfrac{1}{4}(\delta^\mu_\alpha)\hat{\gamma}^{jq} n_{mq}\left[\frac{\partial}{\partial u^\mu}(W^m_\beta W^\beta_j) - \left(\frac{\partial W^m_\beta}{\partial u^\mu}\right) W^\beta_j\right].\end{aligned}$$

Therefore

$$\mathbf{\Gamma}_\alpha = \tfrac{1}{8}\gamma^{\beta\nu}\left(\frac{\partial g_{\nu\beta}}{\partial u^\alpha} + \frac{\partial g_{\nu\alpha}}{\partial u^\beta} - \frac{\partial g_{\alpha\beta}}{\partial u^\nu}\right) - \tfrac{1}{4}\delta^\mu_\alpha W^\beta_j \hat{\gamma}^j \wedge \frac{\partial}{\partial u^\mu}(\hat{\gamma}^q W^m_\beta n_{mq}).$$

Since $\gamma^{\beta\nu} = -\gamma^{\nu\beta}$, it follows that $\gamma^{\beta\nu}(\partial g_{\nu\beta}/\partial u^\alpha) = 0$. Thus we have

$$\mathbf{\Gamma}_\alpha = \tfrac{1}{8}\gamma^{\beta\nu}\left(\frac{\partial g_{\nu\alpha}}{\partial u^\beta} - \frac{\partial g_{\alpha\beta}}{\partial u^\nu}\right) - \tfrac{1}{4}\gamma^\beta \wedge \frac{\partial}{\partial u^\alpha}\gamma_\beta,$$

and finally

$$\mathbf{\Gamma}_\alpha = \tfrac{1}{4}\gamma^{\beta\nu}\frac{\partial g_{\alpha\nu}}{\partial u^\beta} - \tfrac{1}{4}\gamma^\beta \wedge \frac{\partial}{\partial u^\alpha}\gamma_\beta. \tag{5.42}$$

In the context of general relativity, the orthonormal frame is known as a *Vierbein.* The definition of $\partial/\partial u^\alpha$ and therefore $\mathbf{\Gamma}_\alpha$ must depend on the choice of the Dirac system of orthonormal matrices or Vierbein. The first term on the right hand side of Eq. (5.42) is independent of the choice of orthonormal frame. Sometimes it is possible to choose an orthonormal frame that will make the second term zero. This is possible in the case of the Schwarzschild and Kerr metrics.

If the metric tensor for the coordinate system is diagonal, then one can simplify this formula even more. In particular the first term is simplified and the second term becomes zero if one chooses the constant orthonormal frame to be parallel to the coordinate frame. In particular, we can let

$$\gamma_\beta = \hat{\gamma}_b |g_{\beta\beta}|^{\frac{1}{2}} = g_{\beta\beta}\gamma^\beta$$

where $\beta = b = 1, 2, \ldots,$ or n and no sum is intended. Then

$$\begin{aligned}\frac{\partial}{\partial u^\alpha}\gamma_\beta &= \hat{\gamma}_b \tfrac{1}{2}|g_{\beta\beta}|^{-\frac{1}{2}}\frac{\partial |g_{\beta\beta}|}{\partial u^\alpha} \\ &= \tfrac{1}{2}\gamma^\beta \frac{g_{\beta\beta}}{|g_{\beta\beta}|}\frac{\partial |g_{\beta\beta}|}{\partial u^\alpha} \\ &= \tfrac{1}{2}\gamma^\beta \frac{\partial g_{\beta\beta}}{\partial u^\alpha}.\end{aligned}$$

It then follows that

$$\gamma^\beta \wedge \frac{\partial}{\partial u^\alpha}\gamma_\beta = \tfrac{1}{2}(\gamma^\beta \wedge \gamma^\beta)\frac{\partial g_{\beta\beta}}{\partial u^\alpha} = \mathbf{0}.$$

For the first term in Eq. (5.42), we merely note that in the summation over the ν index $g_{\nu\alpha} = 0$ unless $\nu = \alpha$. Thus if $g_{\alpha\beta}$ is diagonal, we may write

$$\mathbf{\Gamma}_\alpha = \tfrac{1}{4}\gamma^{\beta\alpha}\frac{\partial}{\partial u^\beta}g_{\alpha\alpha} \tag{5.43}$$

where the β is summed but the α index is not.

To generalize Eq. (5.40), we first note that

$$\begin{aligned}\nabla_\alpha(\hat{\gamma}^k) &= \nabla_\alpha(n^{kj}\hat{\gamma}_j) = n^{kj}\nabla_\alpha\hat{\gamma}_j\\ &= n^{kj}(-\mathbf{\Gamma}_\alpha\hat{\gamma}_j + \hat{\gamma}_j\mathbf{\Gamma}_\alpha)\end{aligned}$$

and thus

$$\nabla_\alpha(\hat{\gamma}^k) = -\mathbf{\Gamma}_\alpha\hat{\gamma}^k + \hat{\gamma}^k\mathbf{\Gamma}_\alpha.$$

For a product of orthonormal Dirac matrices, we get some nifty cancellations. For example

$$\begin{aligned}\nabla_\alpha(\hat{\gamma}^k\hat{\gamma}_j\hat{\gamma}_p) &= (\nabla_\alpha\hat{\gamma}^k)\hat{\gamma}_j\hat{\gamma}_p + \hat{\gamma}^k(\nabla_\alpha\hat{\gamma}_j)\hat{\gamma}_p + \hat{\gamma}^k\hat{\gamma}_j(\nabla_\alpha\hat{\gamma}_p)\\ &= (-\mathbf{\Gamma}_\alpha\hat{\gamma}^k\hat{\gamma}_j\hat{\gamma}_p + \hat{\gamma}^k\mathbf{\Gamma}_\alpha\hat{\gamma}_j\hat{\gamma}_p)\\ &\quad + (-\hat{\gamma}^k\mathbf{\Gamma}_\alpha\hat{\gamma}_j\hat{\gamma}_p + \hat{\gamma}^k\hat{\gamma}_j\mathbf{\Gamma}_\alpha\hat{\gamma}_p)\\ &\quad + (-\hat{\gamma}^k\hat{\gamma}_j\mathbf{\Gamma}_\alpha\hat{\gamma}_p + \hat{\gamma}^k\hat{\gamma}_j\hat{\gamma}_p\mathbf{\Gamma}_p\mathbf{\Gamma}_\alpha),\end{aligned}$$

and thus

$$\nabla_\alpha(\hat{\gamma}^k\hat{\gamma}_j\hat{\gamma}_p) = -\mathbf{\Gamma}_\alpha(\hat{\gamma}^k\hat{\gamma}_j\hat{\gamma}_p) + (\hat{\gamma}^k\hat{\gamma}_j\hat{\gamma}_p)\mathbf{\Gamma}_\alpha.$$

For a Clifford number of the form $A_{ki}{}^s\hat{\gamma}^k\hat{\gamma}_j\hat{\gamma}_p$, we would have

$$\nabla_\alpha(A_{ki}{}^s\hat{\gamma}^k\hat{\gamma}_j\hat{\gamma}_p) = \frac{\partial A_{ki}{}^s}{\partial u^\alpha}\hat{\gamma}^k\hat{\gamma}_j\hat{\gamma}_p - \mathbf{\Gamma}_\alpha(A_{ki}{}^s\hat{\gamma}^k\hat{\gamma}_j\hat{\gamma}_p) + (A_{ki}{}^s\hat{\gamma}^k\hat{\gamma}_j\hat{\gamma}_p)\mathbf{\Gamma}_\alpha.$$

From these illustrative examples, one can see that

$$\nabla_\alpha[\quad] = \frac{\partial}{\partial u^\alpha}[\quad] - \mathbf{\Gamma}_\alpha[\quad] + [\quad]\mathbf{\Gamma}_\alpha \tag{5.44}$$

where any differentiable Clifford number can be inserted in the square brackets.

If one wants to use some directional derivatives associated with some noncoordinate frame, one can define

$$\Gamma_a = W_a^\alpha \Gamma_\alpha. \tag{5.45}$$

Multiplying Eq. (5.44) by W_a^α and summing over the α index then gives us

$$\nabla_a[\quad] = \partial_a[\quad] - \Gamma_a[\quad] + [\quad]\Gamma_a. \tag{5.46}$$

A comment should be made here. All equations and remarks that were made for Fock–Ivanenko 2-vectors using an orthonormal system of Dirac matrices for which $\hat{\gamma}_{k,\alpha} = 0$, remain valid for any constant frame. This point was not made in the above text because of the limitations of the notation that I have selected for this book.

In closing this section, I would like to express the opinion that use of Fock–Ivanenko 2-vectors usually provides the most efficient method of computing the Riemann curvature tensor. This point will be pursued in Section 5.4.

Problem 5.6. For the 2-dimensional surface of a sphere, $g_{\theta\theta} = r^2$, $g_{\phi\phi} = r^2 \sin\theta$. Use Eq. (5.43) to compute Γ_θ and Γ_ϕ.

Problem 5.7. Use Eq. (5.36) to show that

$$c_{ijk} = \Gamma_{ijk} - \Gamma_{jik}. \tag{5.47}$$

Problem 5.8. Define $[\partial_i, \partial_j] = \partial_i \partial_j - \partial_j \partial_i$.

(1) Show

$$[[\partial_i, \partial_j], \partial_k] + [[\partial_j, \partial_k], \partial_i] + [[\partial_k, \partial_i], \partial_j] = 0. \tag{5.48}$$

(2) Use Eqs. (5.31) and (5.48) to show that

$$c_{ij}{}^p c_{pk}{}^m + c_{jk}{}^p c_{pi}{}^m + c_{ki}{}^p c_{pj}{}^m = 0. \tag{5.49}$$

(Equations (5.48) and (5.49) are known as *Jacobi identities.*)

5.3 The Riemann Curvature Tensor and its Symmetries

For a coordinate system of Dirac matrices, the *Riemann curvature tensor* $R_{\alpha\beta\nu}{}^\eta$ may be defined by the relation

$$(\nabla_\alpha \nabla_\beta - \nabla_\beta \nabla_\alpha)\gamma_\nu = R_{\alpha\beta\nu}{}^\eta \gamma_\eta. \tag{5.50}$$

(This is the convention used in *Spinors and Space-Time*, by Roger Penrose and Wolfgang Rindler (1984, p. 200). However, it is not standard for the simple reason that there is no standard notation. Virtually every conceivable variation has been and is used.)

Because of the ∇_μ's that appear on the left-hand side of Eq. (5.50), it is surprising that $R_{\alpha\beta\nu}{}^{\eta}$ is a tensor. However, this entity does indeed behave as a tensor under a change of coordinates. (See Problem 5.9.) On the other hand, Eq. (5.50) must be adjusted for noncoordinate systems. If we require that

$$R_{ijk}{}^{m} = W_i^\alpha W_j^\beta W_k^\nu W_\eta^m R_{\alpha\beta\nu}{}^{\eta}, \tag{5.51}$$

we then find that $R_{ijk}{}^{m}$ must satisfy the equation

$$(\nabla_i \nabla_j - \nabla_j \nabla_i)\gamma_k - c_{ij}{}^{m}\nabla_m \gamma_k = R_{ijk}{}^{m}\gamma_m \tag{5.52}$$

where the $c_{ij}{}^{m}$'s are the commutator coefficients defined by Eqs. (5.31) and (5.32).

It is useful to state and then prove some symmetries for the indices of R_{ijkm}. First, R_{ijkm} is antisymmetric with respect to its first two indices. That is

$$R_{ijkm} = -R_{jikm}. \tag{5.53}$$

The tensor is also antisymmetric with respect to its last two indices. That is

$$R_{ijkm} = -R_{ijmk}. \tag{5.54}$$

Furthermore, there is a cyclic symmetry in the first three indices. In particular

$$R_{ijkm} + R_{jkim} + R_{kijm} = 0. \tag{5.55}$$

(This is generalized in Prob. 5.12.) Finally, the curvature tensor is symmetric with respect to an exchange of the first pair of indices with the second pair:

$$R_{ijkm} = R_{kmij}. \tag{5.56}$$

We will now verify these symmetries for coordinate systems and then note that the same symmetries immediately follow for noncoordinate frames since

$$R_{ijkm} = W_i^\alpha W_j^\beta W_k^\nu W_m^\eta R_{\alpha\beta\nu\eta}. \tag{5.57}$$

To verify that $R_{\alpha\beta\nu\eta} = -R_{\beta\alpha\nu\eta}$, we merely reexamine Eq. (5.50) which was used to define the tensor.

To show that $R_{\alpha\beta\nu\eta} = -R_{\alpha\beta\eta\nu}$ is more difficult. To obtain this result, we

make use of the fact that ∇_α acts on $g_{\nu\eta}$ like a partial derivative. Thus

$$(\nabla_\alpha\nabla_\beta - \nabla_\beta\nabla_\alpha)g_{\nu\eta}I = 0. \tag{5.58}$$

This means that

$$(\nabla_\alpha\nabla_\beta - \nabla_\beta\nabla_\alpha)(\gamma_\nu\gamma_\eta + \gamma_\eta\gamma_\nu) = 0.$$

Applying ∇_β and then ∇_α to the product $\gamma_\nu\gamma_\eta$, we find that

$$\begin{aligned}\nabla_\alpha\nabla_\beta\gamma_\nu\gamma_\eta &= (\nabla_\alpha\nabla_\beta\gamma_\nu)\gamma_\eta + (\nabla_\beta\gamma_\nu)(\nabla_\alpha\gamma_\eta)\\ &\quad + (\nabla_\alpha\gamma_\nu)(\nabla_\beta\gamma_\eta) + \gamma_\nu(\nabla_\alpha\nabla_\beta\gamma_\eta).\end{aligned}$$

A similar equation can be obtained by reversing the α and β indices. Taking the difference of the two equations then gives us

$$\begin{aligned}(\nabla_\alpha\nabla_\beta - \nabla_\beta\nabla_\alpha)\gamma_\nu\gamma_\eta &= ((\nabla_\alpha\nabla_\beta - \nabla_\beta\nabla_\alpha)\gamma_\nu)\,\gamma_\eta\\ &\quad + \gamma_\nu((\nabla_\alpha\nabla_\beta - \nabla_\beta\nabla_\alpha)\gamma_\eta)\\ &= R_{\alpha\beta\nu}{}^{\sigma}\gamma_\sigma\gamma_\eta + R_{\alpha\beta\eta}{}^{\sigma}\gamma_\nu\gamma_\sigma.\end{aligned} \tag{5.59}$$

Now we can reverse the ν and η indices in Eq. (5.59), and then add the resulting equation to Eq. (5.59). One then has

$$(\nabla_\alpha\nabla_\beta - \nabla_\beta\nabla_\alpha)(2g_{\nu\eta}I) = R_{\alpha\beta\nu}{}^{\sigma}(2g_{\sigma\eta}I) + R_{\alpha\beta\eta}{}^{\sigma}(2g_{\nu\sigma}I).$$

Carrying out the summations over the σ index and using Eq. (5.58), one has

$$0 = 2R_{\alpha\beta\nu\eta} + 2R_{\alpha\beta\eta\nu},$$

and this is equivalent to what we set out to prove.

To obtain the cyclic symmetry, we make use of the torsion-free condition. That is $\nabla_\sigma\gamma_\mu = \nabla_\mu\gamma_\sigma$. This implies that

$$\nabla_\alpha\nabla_\beta\gamma_\nu - \nabla_\alpha\nabla_\nu\gamma_\beta = 0,$$

$$\nabla_\beta\nabla_\nu\gamma_\alpha - \nabla_\beta\nabla_\alpha\gamma_\nu = 0,$$

and

$$\nabla_\nu\nabla_\alpha\gamma_\beta - \nabla_\nu\nabla_\beta\gamma_\alpha = 0.$$

Adding these three equations and then regrouping the terms together gives us the equation:

$$(\nabla_\alpha\nabla_\beta - \nabla_\beta\nabla_\alpha)\gamma_\nu + (\nabla_\beta\nabla_\nu - \nabla_\nu\nabla_\beta)\gamma_\alpha + (\nabla_\nu\nabla_\alpha - \nabla_\alpha\nabla_\nu)\gamma_\beta = 0,$$

or

$$(R_{\alpha\beta\nu}{}^{\sigma} + R_{\beta\nu\alpha}{}^{\sigma} + R_{\nu\alpha\beta}{}^{\sigma})\gamma_{\sigma} = 0. \tag{5.60}$$

Multiplying both sides of Eq. (5.60) by γ_{η} from the right gives us a new equation. Multiplying Eq. (5.60) by γ_{η} from the left gives us a second new equation. Adding the resulting equations gives us

$$(R_{\alpha\beta\nu}{}^{\sigma} + R_{\beta\nu\alpha}{}^{\sigma} + R_{\nu\alpha\beta}{}^{\sigma})(2g_{\eta\sigma}I) = 0$$

or

$$R_{\alpha\beta\nu\eta} + R_{\beta\nu\alpha\eta} + R_{\nu\alpha\beta\eta} = 0.$$

To get the last symmetry, we add up four versions of the cyclic symmetry equation. That is

$$\begin{aligned} R_{\alpha\beta\nu\eta} + R_{\beta\nu\alpha\eta} + R_{\nu\alpha\beta\eta} &= 0 \\ R_{\beta\alpha\eta\nu} + R_{\alpha\eta\beta\nu} + R_{\eta\beta\alpha\nu} &= 0 \\ -R_{\nu\eta\alpha\beta} - R_{\eta\alpha\nu\beta} - R_{\alpha\nu\eta\beta} &= 0 \\ -R_{\eta\nu\beta\alpha} - R_{\nu\beta\eta\alpha} - R_{\beta\eta\nu\alpha} &= 0. \end{aligned}$$

When we add these last four equations together, we make use of the fact that $R_{ijkm} = R_{jimk}$. The result of the addition is then

$$2R_{\alpha\beta\nu\eta} - 2R_{\nu\eta\alpha\beta} = 0$$

which is what we set out to prove.

A tensor which is closely associated with the Riemann tensor is the *Ricci tensor* R_{ij} which is defined by the equation

$$R_{ij} = R_{imj}{}^{m}. \tag{5.61}$$

It is not difficult to show that the Ricci tensor is symmetric with respect to its two indices. We merely note that

$$R_{ij} = R_{ipj}{}^{p} = R_{ipjq}g^{qp} = R_{jqip}g^{qp} = R_{jqi}{}^{q} = R_{ji}. \tag{5.62}$$

The Ricci tensor plays a fundamental role in Einstein's theory of general relativity. This point will be discussed in Chapter 6.

Problem 5.9. Note that

$$(\bar{\nabla}_{\alpha}\bar{\nabla}_{\beta} - \bar{\nabla}_{\beta}\bar{\nabla}_{\alpha})\bar{\gamma}_{\nu} = \left[\left(\frac{\partial u^{\delta}}{\partial \bar{u}^{\alpha}}\nabla_{\delta}\right)\left(\frac{\partial u^{\sigma}}{\partial \bar{u}^{\beta}}\nabla_{\sigma}\right) - \left(\frac{\partial u^{\sigma}}{\partial \bar{u}^{\beta}}\nabla_{\sigma}\right)\left(\frac{\partial u^{\delta}}{\partial \bar{u}^{\alpha}}\nabla_{\delta}\right)\right]\left(\frac{\partial u^{\mu}}{\partial \bar{u}^{\nu}}\gamma_{\mu}\right).$$

(1) Use this relation to show that

$$(\bar{\nabla}_\alpha \bar{\nabla}_\beta - \bar{\nabla}_\beta \bar{\nabla}_\alpha)\bar{\gamma}_\nu = \frac{\partial u^\delta}{\partial \bar{u}^\alpha}\frac{\partial u^\sigma}{\partial \bar{u}^\beta}\frac{\partial u^\mu}{\partial \bar{u}^\nu}(\nabla_\delta \nabla_\sigma - \nabla_\sigma \nabla_\delta)\gamma_\mu.$$

(2) Use the result from part (1) to show that $R_{\alpha\beta\nu}{}^\eta$ transforms as a tensor under a change of coordinates.

Problem 5.10.

(1) Show that

$$W_i^\alpha W_j^\beta W_k^\nu(\nabla_\alpha \nabla_\beta - \nabla_\beta \nabla_\alpha)\gamma_\nu = (\nabla_i \nabla_j - \nabla_j \nabla_i)\gamma_k - c_{ij}{}^m \nabla_m \gamma_k$$

where the $c_{ij}{}^m$'s are the commutator coefficients defined by Eqs. (5.31) and (5.32). Suggestion: first show that $(\nabla_\alpha \nabla_\beta - \nabla_\beta \nabla_\alpha)(W_k^\nu \gamma_\nu) = W_k^\nu(\nabla_\alpha \nabla_\beta - \nabla_\beta \nabla_\alpha)\gamma_\nu$.

(2) Using the result from part (1), show that if

$$R_{ijk}{}^m = W_i^\alpha W_j^\beta W_k^\nu W_\eta^m R_{\alpha\beta\nu}{}^\eta,$$

then

$$R_{ijk}{}^m \gamma_m = (\nabla_i \nabla_j - \nabla_j \nabla_i)\gamma_k - c_{ij}{}^m \nabla_m \gamma_k.$$

Problem 5.11. Use the relation that $R_{\alpha\beta\nu\eta} = R_{\nu\eta\alpha\beta}$ and Eq. (5.57) to show that $R_{ijkm} = R_{kmij}$.

Problem 5.12. In this section, it was shown that the Riemann tensor has a cyclic symmetry in the first three indices. Show that the Riemann tensor has the same cyclic symmetry in any of its three indices. For example show $R_{ijkm} + R_{ikmj} + R_{imjk} = 0$.

Problem 5.13.

(1) Use the Leibniz property of ∇_μ to show

$$(\nabla_\alpha \nabla_\beta - \nabla_\beta \nabla_\alpha)\gamma_\eta \gamma^\nu = [(\nabla_\alpha \nabla_\beta - \nabla_\beta \nabla_\alpha)\gamma_\eta]\gamma^\nu + \gamma_\eta(\nabla_\alpha \nabla_\beta - \nabla_\beta \nabla_\alpha)\gamma^\nu.$$

(2) Using the fact that $(\nabla_\alpha \nabla_\beta - \nabla_\beta \nabla_\alpha)\delta_\eta^\nu I = 0$, demonstrate that

$$(\nabla_\alpha \nabla_\beta - \nabla_\beta \nabla_\alpha)\gamma^\nu = -R_{\alpha\beta\eta}{}^\nu \gamma^\eta.$$

(3) Show

$$(\nabla_\alpha \nabla_\beta - \nabla_\beta \nabla_\alpha)A^{\eta\sigma}{}_\nu \gamma_\eta \gamma_\sigma \gamma^\nu$$
$$= (A^{\lambda\sigma}{}_\nu R_{\alpha\beta\lambda}{}^\eta + A^{\eta\lambda}{}_\nu R_{\alpha\beta\lambda}{}^\sigma - A^{\eta\sigma}{}_\lambda R_{\alpha\beta\gamma}{}^\lambda)\gamma_\eta \gamma_\sigma \gamma^\nu.$$

Problem 5.14. Use the contravariant version of Eq. (3.31) to show that $R_{ijkm}\gamma^i\gamma^j\gamma^k = -2R_{pm}\gamma^p$.

Problem 5.15. Use the contravariant version of Eq. (3.32) to show that $R_{ijkm}\gamma^i\gamma^j\gamma^k\gamma^m = -2RI$ where $R = R_j{}^j = R_{jk}g^{jk}$.

Problem 5.16. Use Eq. (5.52) and the relation $\nabla_i\gamma_j = \Gamma_{ij}{}^m\gamma_m$ to show that

$$R_{ijk}{}^m = \partial_i\Gamma_{jk}{}^m - \partial_j\Gamma_{ik}{}^m + \Gamma_{jk}{}^p\Gamma_{ip}{}^m - \Gamma_{ik}{}^p\Gamma_{jp}{}^m - c_{ij}{}^p\Gamma_{pk}{}^m. \tag{5.63}$$

Problem 5.17. Use Eqs. (5.47) and (5.63) to show that

$$R_{ijkm} = \partial_i\Gamma_{jkm} - \partial_j\Gamma_{ikm} + \Gamma_{jk}{}^p\Gamma_{ipm} - \Gamma_{ik}{}^p\Gamma_{jpm} - \Gamma_{ij}{}^p\Gamma_{pkm} + \Gamma_{ji}{}^p\Gamma_{pkm}. \tag{5.64}$$

Problem 5.18. Show that

$$A^j{}_{;k;i} - A^j{}_{;i;k} = R_{ikp}{}^jA^p. \tag{5.65}$$

5.4 The Use of Fock–Ivanenko 2-Vectors to Compute Curvature 2-Forms

In Section 5.2, we were able to exploit the antisymmetry of the last two indices of the Ricci rotation coefficients. In particular that antisymmetry enabled us to express all differentiation relations for Clifford numbers in terms of Fock–Ivanenko 2-vectors. It is possible to exploit the antisymmetry of the last two indices of the totally covariant version of the Riemann curvature tensor in a similar way.

With that intent, we introduce the notion of a *curvature 2-form.* The symbol $\mathscr{R}_{ij}$ will be used to designate a curvature 2-form where

$$\mathscr{R}_{ij} = \tfrac{1}{2}R_{ijkm}\gamma^{km}. \tag{5.66}$$

It is a slight abuse of terminology to refer to $\mathscr{R}_{ij}$ as a 2-form rather than a 2-vector. However, it is the Clifford algebra analogue of the 2-form that appears in the theory of differential forms.

For Fock–Ivanenko 2-vectors, it was shown that

$$\Gamma_{ij}{}^k\hat{\gamma}_k = \Gamma_{ijk}\hat{\gamma}^k = -\mathbf{\Gamma}_i\hat{\gamma}_k + \hat{\gamma}_k\mathbf{\Gamma}_i$$

and thus

$$\nabla_i\hat{\gamma}_k = -\mathbf{\Gamma}_i\hat{\gamma}_k + \hat{\gamma}_k\mathbf{\Gamma}_i.$$

In a similar fashion, it can be shown that

$$R_{ijk}{}^m\gamma_m = R_{ijkm}\gamma^m = -\tfrac{1}{2}\mathscr{R}_{ij}\gamma_k + \gamma_k(\tfrac{1}{2}\mathscr{R}_{ij}) \tag{5.67}$$

and thus

$$(\nabla_i\nabla_j - \nabla_j\nabla_i)\gamma_k - c_{ij}{}^m\nabla_m\gamma_k = -\tfrac{1}{2}\mathscr{R}_{ij}\gamma_k + \gamma_k(\tfrac{1}{2}\mathscr{R}_{ij}). \tag{5.68}$$

It is left to the reader to show that to verify Eq. (5.67), one merely uses Eq. (5.41) in much the same way as it was used to verify Eq. (5.40) for Fock–Ivanenko 2-vectors.

Equation (5.68) may be generalized in essentially the same way that Eq. (5.40) was generalized for Fock–Ivanenko 2-vectors to obtain Eq. (5.43). The only significant difference is that a tensor coefficient of a p-vector commutes with the curvature operator. For example

$$(\nabla_i\nabla_j - \nabla_j\nabla_i - c_{ij}{}^m\nabla_m)A_{pq}{}^s\gamma^p\gamma_r\gamma_t = A_{pq}{}^s(\nabla_i\nabla_j - \nabla_j\nabla_i - c_{ij}{}^m\nabla_m)\gamma^p\gamma_r\gamma_t. \tag{5.69}$$

In this fashion the generalized version of Eq. (5.68) becomes

$$(\nabla_i\nabla_j - \nabla_j\nabla_i - c_{ij}{}^m\nabla_m)[\quad] = -\tfrac{1}{2}\mathscr{R}_{ij}[\quad] + [\quad](\tfrac{1}{2}\mathscr{R}_{ij}) \tag{5.70}$$

where any twice differentiable Clifford number can be inserted in the square brackets.

Perhaps one should note that if a tensor multiple of I were added to $\mathscr{R}_{ij}$, Eq. (5.70) would remain valid. On the other hand, if we require $\mathscr{R}_{ij}$ to be a 2-vector, it is unique.

It is now not too difficult to obtain a simple formula for curvature 2-forms in terms of Fock–Ivanenko 2-vectors. Using the fact that

$$\nabla_p\gamma_k = -\boldsymbol{\Gamma}_p\gamma_k + \gamma_k\boldsymbol{\Gamma}_p,$$

one can show that

$$\begin{aligned}(\nabla_i\nabla_j - \nabla_j\nabla_i - c_{ij}{}^m\nabla_m)\gamma_k = &-(\nabla_i\boldsymbol{\Gamma}_j - \nabla_j\boldsymbol{\Gamma}_i + \boldsymbol{\Gamma}_i\boldsymbol{\Gamma}_j - \boldsymbol{\Gamma}_j\boldsymbol{\Gamma}_i - c_{ij}{}^m\boldsymbol{\Gamma}_m)\gamma_k\\ &+ \gamma_k(\nabla_i\boldsymbol{\Gamma}_j - \nabla_j\boldsymbol{\Gamma}_i + \boldsymbol{\Gamma}_i\boldsymbol{\Gamma}_j - \boldsymbol{\Gamma}_j\boldsymbol{\Gamma}_i - c_{ij}{}^m\boldsymbol{\Gamma}_m).\end{aligned}$$

Comparing this result with Eq. (5.70), one would guess that

$$\tfrac{1}{2}\mathscr{R}_{ij} = \nabla_i\boldsymbol{\Gamma}_j - \nabla_j\boldsymbol{\Gamma}_i + \boldsymbol{\Gamma}_i\boldsymbol{\Gamma}_j - \boldsymbol{\Gamma}_j\boldsymbol{\Gamma}_i - c_{ij}{}^m\boldsymbol{\Gamma}_m. \tag{5.71}$$

The only thing left to verify is that the right-hand side of Eq. (5.71) is a 2-vector. This is left to the reader. (See Problem 5.19.)

An alternate form of Eq. (5.71) may be obtained by noting that

$$\begin{aligned}\nabla_i\boldsymbol{\Gamma}_j &= W_i^\alpha \frac{\partial}{\partial u^\alpha}\boldsymbol{\Gamma}_j - \boldsymbol{\Gamma}_i\boldsymbol{\Gamma}_j + \boldsymbol{\Gamma}_j\boldsymbol{\Gamma}_i\\ &= \partial_i\boldsymbol{\Gamma}_j - \boldsymbol{\Gamma}_i\boldsymbol{\Gamma}_j + \boldsymbol{\Gamma}_j\boldsymbol{\Gamma}_i;\end{aligned}$$

we then get

$$\tfrac{1}{2}\mathscr{R}_{ij} = \partial_i \mathbf{\Gamma}_j - \partial_j \mathbf{\Gamma}_i - \mathbf{\Gamma}_i \mathbf{\Gamma}_j + \mathbf{\Gamma}_j \mathbf{\Gamma}_i - c_{ij}{}^m \mathbf{\Gamma}_m. \tag{5.72}$$

(Comparing Eq. (5.71) and (5.72), it is important to observe the difference in signs that occurs in front of the term $(\mathbf{\Gamma}_i \mathbf{\Gamma}_j - \mathbf{\Gamma}_j \mathbf{\Gamma}_i)$.)

Equations (5.71) and (5.72) are probably most useful for coordinate systems for which the commutator coefficients are zero. In the case of the 4-dimensional space-time manifold of general relativity one has only four Fock–Ivanenko 2-vectors to deal with in place of forty Christoffel symbols. This fact will be exploited in Chapter 6 to compute the Schwarzschild metric.

Problem 5.19.

(1) Use the covariant version of Eq. (3.34) to show that

$$\gamma^{pq}\gamma^{rs} = \gamma^{pqrs} - g^{pr}\gamma^{qs} + g^{ps}\gamma^{qr} + g^{qr}\gamma^{ps} - g^{qs}\gamma^{pr} + g^{ps}g^{qr}.$$

(2) Use the result of part (1) to show that

$$\gamma^{pq}\gamma^{rs} - \gamma^{rs}\gamma^{pq} = -2g^{pr}\gamma^{qs} + 2g^{ps}\gamma^{qr} + 2g^{qr}\gamma^{ps} - 2g^{qs}\gamma^{pr}.$$

(3) Use the result of part (2) to show that

$$\mathbf{\Gamma}_i \mathbf{\Gamma}_j - \mathbf{\Gamma}_j \mathbf{\Gamma}_i \text{ is a 2-vector.}$$

Problem 5.20. Use Eq. (5.72) and the result of Problem 5.6 to compute the curvature 2-form $\tfrac{1}{2}\mathscr{R}_{\theta\phi}$ for the 2-dimensional surface of the sphere.

Problem 5.21. A metric important in the study of cosmology is the *Friedman metric* defined by the equation:

$$\mathrm{d}s^2 = \mathrm{d}\tau^2 - (a(\tau))^2[\mathrm{d}\chi^2 + \sin^2\chi(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)].$$

Use Eq. (5.43) to show that

$$\mathbf{\Gamma}_\tau = 0, \qquad \mathbf{\Gamma}_\chi = \tfrac{1}{2}\gamma^{\chi\tau}a(\tau)a'(\tau),$$
$$\mathbf{\Gamma}_\theta = \tfrac{1}{2}\gamma^{\theta\tau}aa'\sin^2\chi - \tfrac{1}{2}\gamma^{\chi\theta}a^2\sin\chi\cos\chi$$

and

$$\begin{aligned}\mathbf{\Gamma}_\phi = {} & \tfrac{1}{2}\gamma^{\phi\tau}aa'\sin^2\chi\sin^2\theta \\ & + \tfrac{1}{2}\gamma^{\phi\chi}a^2\sin\chi\cos\chi\sin^2\theta \\ & - \tfrac{1}{2}\gamma^{\theta\phi}a^2\sin^2\chi\sin\theta\cos\theta.\end{aligned}$$

Problem 5.22. Use the results of Problem 5.21 and Eq. (5.72) to show that for the Friedman metric, the curvature 2-forms are

$$\begin{aligned}
\mathscr{R}_{\chi\tau} &= -\gamma^{\chi\tau}aa'', \qquad \mathscr{R}_{\theta\tau} = -\gamma^{\theta\tau}aa''\sin^2\chi, \\
\mathscr{R}_{\phi\tau} &= -\gamma^{\phi\tau}aa''\sin^2\chi\sin^2\theta, \\
\mathscr{R}_{\theta\phi} &= \gamma^{\theta\phi}(1+(a')^2)a^2\sin^4\chi\sin^2\theta, \\
\mathscr{R}_{\phi\chi} &= \gamma^{\phi\chi}(1+(a')^2)a^2\sin^2\chi\sin^2\theta, \\
\mathscr{R}_{\chi\theta} &= \gamma^{\chi\theta}(1+(a')^2)a^2\sin^2\chi.
\end{aligned}$$

Problem 5.23. An important class of metric tensors can be characterized by the relation $\gamma_j = \hat{\gamma}_j - mw_jw^k\hat{\gamma}_k$ where $w^k = n^{kp}w_p$, $w_jw_kn^{jk} = 0$, $g_{jk} = \langle\gamma_j, \gamma_k\rangle$, $n_{jk} = \langle\hat{\gamma}_j, \hat{\gamma}_k\rangle$, and m is an arbitrary constant. A metric tensor of this form is said to be *degenerate.*

(1) Show that $g_{jk} = n_{jk} - 2mw_jw_k$.
(2) Show that the matrix $g^{jk} = n^{jk} + 2mw^jw^k$ is the inverse of g_{jk}.
(3) Show that $g^{jp}w_p = n^{jp}w_p = w^j$.
(4) Use Eq. (5.42) to show that

$$\Gamma_i = -\frac{m}{2}\gamma^{jk}\frac{\partial}{\partial u^j}(w_iw_k).$$

5.5 The Interpretation of Curvature 2-Forms as Infinitesimal Rotation Operators

In the last section of Chapter 4, an index free Clifford number $\mathscr{F}$ was said to be "parallel transported" along a path $u^\alpha(s)$ if it satisfied the equation

$$\nabla_s\mathscr{F} = \frac{du^\alpha}{ds}\nabla_\alpha\mathscr{F} = 0. \tag{5.73}$$

As an index free Clifford number is parallel transported along a path, it will maintain a constant magnitude. An observer moving along the path with a parallel transported vector would observe no rotation. However, in general the vector would undergo a rotation with respect to the coordinate frame. Or perhaps one should say that the coordinate frame rotates with respect to the parallel transported vector. At any rate this relative motion may be described by a rotation operator, that is

$$\mathscr{F}(s) = \mathscr{R}(s)\mathscr{F}(0)\mathscr{R}^{-1}(s). \tag{5.74}$$

This implies that

$$\begin{aligned}\frac{\mathrm{d}\mathscr{F}(s)}{\mathrm{d}s} &= \left(\frac{\mathrm{d}\mathscr{R}(s)}{\mathrm{d}s}\right)\mathscr{F}(0)\mathscr{R}^{-1}(s) + \mathscr{R}(s)\mathscr{F}(0)\frac{\mathrm{d}\mathscr{R}^{-1}(s)}{\mathrm{d}s} \\ &= \left(\frac{\mathrm{d}\mathscr{R}(s)}{\mathrm{d}s}\right)\mathscr{R}^{-1}(s)(\mathscr{R}(s)\mathscr{F}(0)\mathscr{R}^{-1}(s)) \\ &\quad + (\mathscr{R}(s)\mathscr{F}(0)\mathscr{R}^{-1}(s))\mathscr{R}(s)\frac{\mathrm{d}\mathscr{R}^{-1}(s)}{\mathrm{d}s}.\end{aligned}$$

Therefore

$$\frac{\mathrm{d}\mathscr{F}(s)}{\mathrm{d}s} = \left(\frac{\mathrm{d}\mathscr{R}(s)}{\mathrm{d}s}\right)\mathscr{R}^{-1}(s)\mathscr{F}(s) + \mathscr{F}(s)\mathscr{R}(s)\frac{\mathrm{d}\mathscr{R}^{-1}(s)}{\mathrm{d}s}. \tag{5.75}$$

But

$$\mathbf{0} = \frac{d}{ds}(\mathscr{R}(s)\mathscr{R}^{-1}(s)) = \frac{\mathrm{d}\mathscr{R}(s)}{\mathrm{d}s}\mathscr{R}^{-1}(s) + \mathscr{R}(s)\frac{\mathrm{d}\mathscr{R}^{-1}(s)}{\mathrm{d}s}.$$

So we have

$$\frac{\mathrm{d}\mathscr{F}(s)}{\mathrm{d}s} = \left(\frac{\mathrm{d}\mathscr{R}(s)}{\mathrm{d}s}\right)\mathscr{R}^{-1}(s)\mathscr{F}(s) - \mathscr{F}(s)\left(\frac{\mathrm{d}\mathscr{R}(s)}{\mathrm{d}s}\right)\mathscr{R}^{-1}(s) \tag{5.76}$$

On the other hand, from Eq. (5.73)

$$\mathbf{0} = \frac{du^\alpha}{\mathrm{d}s}\nabla_\alpha\mathscr{F} = \frac{du^\alpha}{\mathrm{d}s}\frac{\partial\mathscr{F}}{\partial u^\alpha} - \frac{du^\alpha}{\mathrm{d}s}\mathbf{\Gamma}_\alpha\mathscr{F} + \mathscr{F}\frac{du^\alpha}{\mathrm{d}s}\mathbf{\Gamma}_\alpha$$

or

$$\frac{\mathrm{d}\mathscr{F}}{\mathrm{d}s} = \frac{du^\alpha}{\mathrm{d}s}\mathbf{\Gamma}_\alpha\mathscr{F} - \mathscr{F}\frac{du^\alpha}{\mathrm{d}s}\mathbf{\Gamma}_\alpha. \tag{5.77}$$

Comparing Eqs. (5.76) and (5.77), we have

$$\frac{\mathrm{d}\mathscr{R}(s)}{\mathrm{d}s} = \frac{\mathrm{d}u^\nu}{\mathrm{d}s}\mathbf{\Gamma}_\nu\mathscr{R}(s). \tag{5.78}$$

For small displacements

$$\mathscr{R}(\Delta s) = \left(\mathscr{R}(s) + \Delta s\frac{\mathrm{d}\mathscr{R}(s)}{\mathrm{d}s} + \tfrac{1}{2}(\Delta s)^2\frac{\mathrm{d}^2\mathscr{R}(s)}{\mathrm{d}s^2}\right)\bigg|_{s=0}. \tag{5.79}$$

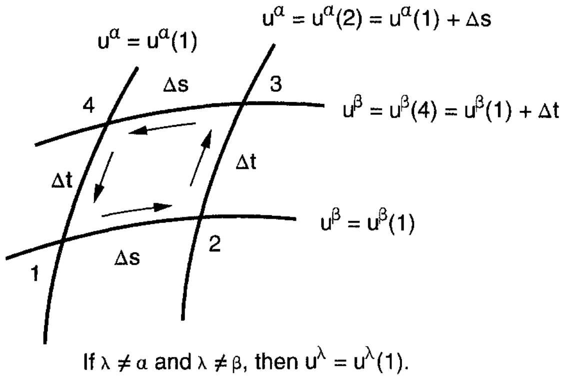


Fig. 5.1. A coordinate loop in the surface obtained by setting all coordinates equal to various constants except u^α and u^β.

We note that $\mathscr{R}(0) = I$ and so from Eq. (5.78)

$$\left.\frac{\mathrm{d}\mathscr{R}(s)}{\mathrm{d}s}\right|_{s=0} = \frac{\mathrm{d}u^\nu(0)}{\mathrm{d}s}\boldsymbol{\Gamma}_\nu(0).$$

Also from Eq. (5.78) it follows that

$$\frac{\mathrm{d}^2\mathscr{R}(s)}{\mathrm{d}s^2} = \left(\frac{\mathrm{d}^2u^\nu}{\mathrm{d}s^2}\right)\boldsymbol{\Gamma}_\nu\mathscr{R}(s) + \frac{\mathrm{d}u^\nu}{\mathrm{d}s}\frac{\mathrm{d}\boldsymbol{\Gamma}_\nu}{\mathrm{d}s}\mathscr{R}(s) + \frac{\mathrm{d}u^\nu}{\mathrm{d}s}\boldsymbol{\Gamma}_\nu\frac{\mathrm{d}\mathscr{R}(s)}{\mathrm{d}s}.$$

So

$$\left.\frac{\mathrm{d}^2\mathscr{R}(s)}{\mathrm{d}s^2}\right|_{s=0} = \left(\frac{\mathrm{d}^2u^\nu}{\mathrm{d}s^2}\right)\boldsymbol{\Gamma}_\nu + \frac{\mathrm{d}u^\nu}{\mathrm{d}s}\frac{\mathrm{d}\boldsymbol{\Gamma}_\nu}{\mathrm{d}s} + \frac{\mathrm{d}u^\nu}{\mathrm{d}s}\frac{\mathrm{d}u^\eta}{\mathrm{d}s}\boldsymbol{\Gamma}_\nu\boldsymbol{\Gamma}_\eta$$

where everything on the right-hand side of this last equation is understood to be evaluated at $s = 0$. Equation (5.79) becomes

$$\mathscr{R}(\Delta s) = I + \Delta s\frac{\mathrm{d}u^\nu}{\mathrm{d}s}\boldsymbol{\Gamma}_\nu + \tfrac{1}{2}(\Delta s)^2\left[\left(\frac{\mathrm{d}^2u^\nu}{\mathrm{d}s^2}\right)\boldsymbol{\Gamma}_\nu + \frac{\mathrm{d}u^\nu}{\mathrm{d}s}\frac{\mathrm{d}\boldsymbol{\Gamma}_\nu}{\mathrm{d}s} + \frac{\mathrm{d}u^\nu}{\mathrm{d}s}\frac{\mathrm{d}u^\eta}{\mathrm{d}s}\boldsymbol{\Gamma}_\nu\boldsymbol{\Gamma}_\eta\right] \tag{5.80}$$

where again it is understood that everything is evaluated at $s = 0$.

Now let us consider the result of parallel transporting a Clifford number around a closed path. Consider the path $1 \to 2 \to 3 \to 4 \to 1$ shown in Fig. 5.1. For the first leg of our journey,

$$\frac{\mathrm{d}u^\alpha}{\mathrm{d}s} = 1, \qquad \frac{\mathrm{d}u^\nu}{\mathrm{d}s} = 0 \qquad \text{for } \nu \neq \alpha,$$

and

$$\frac{d\mathbf{\Gamma}_\nu}{ds} = \frac{\partial \mathbf{\Gamma}_\nu}{\partial u^\alpha}$$

and

$$\mathscr{R}_{1\to 2} = I + \Delta s \mathbf{\Gamma}_\alpha(1) + \tfrac{1}{2}(\Delta s)^2 \left[\frac{\partial \mathbf{\Gamma}_\alpha(1)}{\partial u^\alpha} + \mathbf{\Gamma}_\alpha(1)\mathbf{\Gamma}_\alpha(1) \right] \tag{5.81}$$

where the argument "1" is used to indicate the point at which the Fock–Ivanenko 2-vectors are evaluated.

For the second leg of our journey,

$$\frac{du^\beta}{dt} = 1, \qquad \frac{du^\nu}{dt} = 0 \qquad \text{for } \nu \neq \beta, \qquad \frac{d\mathbf{\Gamma}_\nu}{dt} = \frac{\partial \mathbf{\Gamma}_\nu}{\partial u^\beta}.$$

We then have

$$\mathscr{R}_{2\to 3} = I + \Delta t \mathbf{\Gamma}_\beta(2) + \tfrac{1}{2}(\Delta t)^2 \left[\frac{\partial \mathbf{\Gamma}_\beta(2)}{\partial u^\beta} + \mathbf{\Gamma}_\beta(2)\mathbf{\Gamma}_\beta(2) \right]. \tag{5.82}$$

We wish to express this last rotation operator in terms of Fock–Ivanenko 2-vectors evaluated at our point of origination. With this in mind, we note that

$$\mathbf{\Gamma}_\beta(2) = \mathbf{\Gamma}_\beta(1) + \Delta s \frac{d\mathbf{\Gamma}_\beta(1)}{ds} = \mathbf{\Gamma}_\beta(1) + \Delta s \frac{\partial \mathbf{\Gamma}_\beta(1)}{\partial u^\alpha}.$$

Similar expansions can be made for $\partial \mathbf{\Gamma}_\beta(2)/\partial u^\beta$ and $\mathbf{\Gamma}_\beta(2)\mathbf{\Gamma}_\beta(2)$. However, we only need the first term in each of these two expansions to retain a second-order precision in $\mathscr{R}_{2\to 3}$. With these remarks, we now have

$$\mathscr{R}_{2\to 3} = I + \Delta t \mathbf{\Gamma}_\beta(1) + \Delta s \Delta t \frac{\partial \mathbf{\Gamma}_\beta(1)}{\partial u^\alpha} + \tfrac{1}{2}(\Delta t)^2 \left[\frac{\partial \mathbf{\Gamma}_\beta(1)}{\partial u^\beta} + \mathbf{\Gamma}_\beta(1)\mathbf{\Gamma}_\beta(1) \right]. \tag{5.83}$$

To obtain the rotation operator for $\mathscr{R}_{1\to 2\to 3}$, we carry out the obvious multiplication and retain terms up to second order. We then have

$$\begin{aligned} \mathscr{R}_{1\to 2\to 3} = \mathscr{R}_{2\to 3}\mathscr{R}_{1\to 2} &= I + \Delta s \mathbf{\Gamma}_\alpha + \Delta t \mathbf{\Gamma}_\beta + \Delta s \Delta t \left[\frac{\partial \mathbf{\Gamma}_\beta}{\partial u^\alpha} + \mathbf{\Gamma}_\beta \mathbf{\Gamma}_\alpha \right] \\ &\quad + \tfrac{1}{2}(\Delta s)^2 \left[\frac{\partial \mathbf{\Gamma}_\alpha}{\partial u^\alpha} + \mathbf{\Gamma}_\alpha \mathbf{\Gamma}_\alpha \right] + \tfrac{1}{2}(\Delta t)^2 \left[\frac{\partial \mathbf{\Gamma}_\beta}{\partial u} + \mathbf{\Gamma}_\beta \mathbf{\Gamma}_\beta \right] \end{aligned} \tag{5.84}$$

where everything is evaluated at point 1.

We could continue this procedure through point 4 and then on to point

1. But this gets cumbersome. Instead, one can obtain $\mathscr{R}_{1\to 4\to 3}$ by formally switching Δs with Δt and α with β in the formula for $\mathscr{R}_{1\to 2\to 3}$. The result is

$$\mathscr{R}_{1\to 4\to 3} = I + \Delta s\mathbf{\Gamma}_\alpha + \Delta t\mathbf{\Gamma}_\beta + \Delta s\Delta t\left[\frac{\partial\mathbf{\Gamma}_\alpha}{\partial u^\beta} + \mathbf{\Gamma}_\alpha\mathbf{\Gamma}_\beta\right]$$
$$+ \tfrac{1}{2}(\Delta s)^2\left[\frac{\partial\mathbf{\Gamma}_\alpha}{\partial u^\alpha} + \mathbf{\Gamma}_\alpha\mathbf{\Gamma}_\alpha\right] + \tfrac{1}{2}(\Delta t)^2\left[\frac{\partial\mathbf{\Gamma}_\beta}{\partial u^\beta} + \mathbf{\Gamma}_\beta\mathbf{\Gamma}_\beta\right]. \quad (5.85)$$

To obtain the inverse of $\mathscr{R}_{1\to 4\to 3}$, we note that $(I + \delta)^{-1} = 1 - \delta + \delta^2$ where in our case, $\delta =$ all the garbage on the right-hand side of Eq. (5.85) except for I. Carrying out the arithmetic and retaining terms only up to second order, we have

$$\mathscr{R}_{3\to 4\to 1} = I - \Delta s\mathbf{\Gamma}_\alpha - \Delta t\mathbf{\Gamma}_\beta - \Delta s\Delta t\left[\frac{\partial\mathbf{\Gamma}_\alpha}{\partial u^\beta} - \mathbf{\Gamma}_\beta\mathbf{\Gamma}_\alpha\right]$$
$$- \tfrac{1}{2}(\Delta s)^2\left[\frac{\partial\mathbf{\Gamma}_\alpha}{\partial u^\alpha} - \mathbf{\Gamma}_\alpha\mathbf{\Gamma}_\alpha\right] - \tfrac{1}{2}(\Delta t)^2\left[\frac{\partial\mathbf{\Gamma}_\beta}{\partial u^\beta} - \mathbf{\Gamma}_\beta\mathbf{\Gamma}_\beta\right]. \quad (5.86)$$

Finally to get the rotation operator for the entire closed path we have

$$\begin{aligned}\mathscr{R}_{1\to 2\to 3\to 4\to 1} &= \mathscr{R}_{3\to 4\to 1}\mathscr{R}_{1\to 2\to 3}\\ &= I + \Delta s\Delta t\left[\frac{\partial\mathbf{\Gamma}_\beta}{\partial u^\alpha} - \frac{\partial\mathbf{\Gamma}_\alpha}{\partial u^\beta} - \mathbf{\Gamma}_\alpha\mathbf{\Gamma}_\beta + \mathbf{\Gamma}_\beta\mathbf{\Gamma}_\alpha\right]\\ &= I + \Delta s\Delta t(\tfrac{1}{2}\mathscr{R}_{\alpha\beta}). \end{aligned}\quad (5.87)$$

From Eq. (5.87), we see that the curvature 2-form may be interpreted as an infinitesimal rotation operator which acts on a vector or any Clifford number that undergoes parallel transport around a small closed loop formed by segments with coordinate directions.

This result can be generalized to loops consisting of segments with noncoordinate directions.

For example one can first parallel transport a vector from point 1 where $u^\alpha = u^\alpha(1)$ to point 2 along a path defined by the equation

$$\frac{du^\alpha}{ds} = W_i^\alpha \quad \text{for } \alpha = 1, 2, \ldots, n.$$

At point 2, $u^\alpha(2) = u^\alpha(1) + W_i^\alpha(1)\Delta s$. For this first path segment, we get

$$\mathscr{R}_{1\to 2} = I + \Delta s\mathbf{\Gamma}_i(1) + \tfrac{1}{2}(\Delta s)^2\left[\frac{d\mathbf{\Gamma}_i(1)}{ds} + \mathbf{\Gamma}_i(1)\mathbf{\Gamma}_i(1)\right]$$

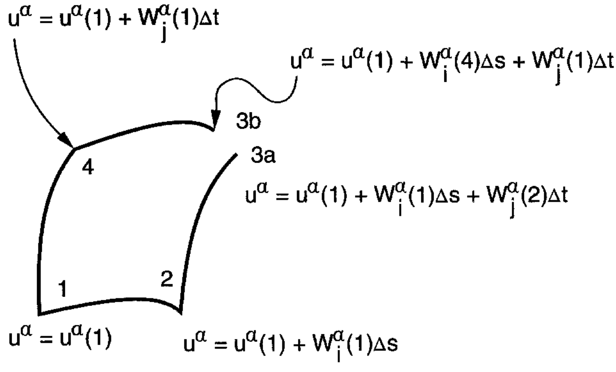


Fig. 5.2. Starting from point 1 and moving first in direction i and then in direction j, one arrives at point 3a. Starting again from point 1 and moving first in direction j and then in direction i, one arrives at point 3b.

where

$$\frac{\mathrm{d}\mathbf{\Gamma}_i(1)}{\mathrm{d}s} = \frac{\mathrm{d}u^\alpha}{\mathrm{d}s}\frac{\partial \mathbf{\Gamma}_i}{\partial u^\alpha} = W_i^\alpha \frac{\partial \mathbf{\Gamma}_i}{\partial u^\alpha} = \partial_i \mathbf{\Gamma}_i$$

The second leg of our journey is governed by the equation

$$\frac{\mathrm{d}u^\alpha}{\mathrm{d}t} = W_j^\alpha .$$

For this leg, we get

$$\mathscr{R}_{2\to 3\mathrm{a}} = I + \Delta t \mathbf{\Gamma}_j(2) + \tfrac{1}{2}(\Delta t)^2 \left[\frac{\mathrm{d}\mathbf{\Gamma}_i(2)}{\mathrm{d}t} + \mathbf{\Gamma}_i(2)\mathbf{\Gamma}_i(2)\right]$$

In a fashion very similar to the coordinate computation, we have

$$\begin{aligned} \mathbf{\Gamma}_j(2) &= \mathbf{\Gamma}_j(1) + \Delta s \frac{\mathrm{d}\mathbf{\Gamma}_j(1)}{\mathrm{d}s} = \mathbf{\Gamma}_j(1) + \Delta s \frac{\mathrm{d}u^\alpha}{\mathrm{d}s}\frac{\partial \mathbf{\Gamma}_j(1)}{\partial u^\alpha} \\ &= \mathbf{\Gamma}_j(1) + \Delta s W_i^\alpha \frac{\partial \mathbf{\Gamma}_j(1)}{\partial u^\alpha} = \mathbf{\Gamma}_j(1) + \Delta s\, \partial_i \mathbf{\Gamma}_j(1). \end{aligned}$$

So far this computation is almost identical to that carried out for a coordinate loop. Now we encounter a difference. At the end of two legs of our journey, one arrives at a point that is labeled by 3a in Fig. 5.2. The coordinates of this point are

$$u^\alpha = u^\alpha(2) + \Delta t W_j^\alpha(2) = u^\alpha(1) + \Delta s W_i^\alpha(1) + \Delta t W_j^\alpha(1) + \Delta s \Delta t\, \partial_i W_j^\alpha(1).$$

When we switch Δs with Δt and i with j, we do not arrive at the same point. We arrive at the point labeled 3b in Fig. 5.2. This point has the coordinates

$$u^\alpha = u^\alpha(1) + \Delta s\, W_i^\alpha(1) + \Delta t\, W_j^\alpha(1) + \Delta s \Delta t\, \partial_j W_i^\alpha(1).$$

It thus becomes necessary to close the path by a short segment where

$$\Delta u^\alpha = \Delta s \Delta t(\partial_j W_i^\alpha - \partial_i W_j^\alpha).$$

From Eq. (5.78),

$$\begin{aligned}\Delta \mathscr{R} = \Delta u^\alpha \mathbf{\Gamma}_\alpha &= \Delta u^\alpha W_\alpha^k \mathbf{\Gamma}_k \\ &= -\Delta s \Delta t\, W_\alpha^k(\partial_i W_j^\alpha - \partial_j W_i^\alpha)\mathbf{\Gamma}_k \\ &= -\Delta s \Delta t c_{ij}{}^k \mathbf{\Gamma}_k.\end{aligned}$$

Thus for this short segment,

$$\mathscr{R}_{3\mathrm{a} \to 3\mathrm{b}} = I - \Delta s \Delta t c_{ij}{}^k \mathbf{\Gamma}_k.$$

With this correction, we get our desired result:

$$\begin{aligned}\mathscr{R}_{1 \to 2 \to 3\mathrm{a} \to 3\mathrm{b} \to 4 \to 1} &= I + \Delta s \Delta t(\partial_i \mathbf{\Gamma}_j - \partial_j \mathbf{\Gamma}_i - \mathbf{\Gamma}_i \mathbf{\Gamma}_j + \mathbf{\Gamma}_j \mathbf{\Gamma}_i - c_{ij}{}^k \mathbf{\Gamma}_k) \\ &= I + \Delta s \Delta t(\tfrac{1}{2}\mathscr{R}_{ij}). \qquad (5.88)\end{aligned}$$

6

THE SCHWARZSCHILD METRIC VIA FOCK–IVANENKO 2-VECTORS

6.1 The Use of Fock–Ivanenko 2-Vectors to Determine the Schwarzschild Metric

About 10 years after his paper on special relativity, Albert Einstein published a series of papers establishing the general theory of relativity (1915a, 1915b, 1916). According to his theory, the path of a small test particle is that of a geodesic in curved space-time. Furthermore the metric for a mass free space must satisfy the equation

$$R_j^{\,k} = 0 \tag{6.1}$$

where $R_j^{\,k}$ is the Ricci tensor. At that time Einstein constructed some approximate solutions for this system which gave credence to his theory. Soon after this, Karl Schwarzschild found an exact solution which is of special interest (1916). For the study of planetary motion about the sun, one is interested in a time-independent spherically symmetric metric. Using various symmetry arguments, Schwarzschild hypothesized that the line element corresponding to such a metric should be of the form

$$(\mathrm{d}s)^2 = f(r)c^2(\mathrm{d}t)^2 - h(r)(\mathrm{d}r)^2 - r^2(\mathrm{d}\theta)^2 - r^2\sin^2\theta(\mathrm{d}\phi)^2. \tag{6.2}$$

In this curved space-time, one can no longer use the same radial coordinate to designate the distance from the center and the circumference of a great circle about the origin divided by 2π. For the Schwarzschild metric, the radial coordinate represents the circumference of a great circle divided by 2π. This is easy to see. On the surface of a sphere with the center at the origin, the line element is

$$(\mathrm{d}s)^2 = -r^2((\mathrm{d}\theta)^2 + \sin^2\theta(\mathrm{d}\phi)^2).$$

This is of course the same as it would be in the flat Lorentz metric. The

circumference of the equatorial circle where $\theta = \pi/2$ and $(\mathrm{d}s)^2 = -r^2(\mathrm{d}\phi)^2$ is

$$\int_0^{2\pi} r\,\mathrm{d}\phi = 2\pi r.$$

On the other hand, to compute the radial distance between two values of r, we note that for $\mathrm{d}t = \mathrm{d}\theta = \mathrm{d}\phi = 0$, we have

$$(\mathrm{d}s)^2 = -h(r)(\mathrm{d}r)^2.$$

Thus the radial distance from r_1 to r_2 is

$$\int_{r_1}^{r_2} (h(r))^{\frac{1}{2}}\,\mathrm{d}r \neq (r_2 - r_1).$$

We now turn to the problem of solving Eq. (6.1) and thereby obtain explicit expressions for $f(r)$ and $h(r)$. From Eq. (6.2),

$$g_{\alpha\beta} = \begin{bmatrix} c^2 f(r) & 0 & 0 & 0 \\ 0 & -h(r) & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2\sin^2\theta \end{bmatrix} \tag{6.3}$$

and

$$g^{\alpha\beta} = \begin{bmatrix} 1/(c^2 f) & 0 & 0 & 0 \\ 0 & -1/h & 0 & 0 \\ 0 & 0 & -1/r^2 & 0 \\ 0 & 0 & 0 & -1/(r^2\sin^2\theta) \end{bmatrix}. \tag{6.4}$$

From these two equations, it is clear how to construct an orthonormal system of Dirac matrices:

$$\gamma^t = \frac{1}{cf^{\frac{1}{2}}}\hat{\gamma}^0 = \frac{1}{c^2 f}\gamma_t, \tag{6.5}$$

$$\gamma^r = \frac{1}{h^{\frac{1}{2}}}\hat{\gamma}^1 = -\frac{1}{h}\gamma_r, \tag{6.6}$$

$$\gamma^\theta = \frac{1}{r}\hat{\gamma}^2 = -\frac{1}{r^2}\gamma_\theta, \tag{6.7}$$

$$\gamma^\phi = \frac{1}{r\sin\theta}\hat{\gamma}^3 = -\frac{1}{r^2\sin^2\theta}\gamma_\phi. \tag{6.8}$$

Since our metric is diagonal, we can use Eq. (5.43) to compute the Fock–Ivanenko 2-vectors. That is

$$\mathbf{\Gamma}_\alpha = \tfrac{1}{4}\gamma^{\beta\alpha}\frac{\partial}{\partial u^\beta} g_{\alpha\alpha}$$

where the index β is summed but α is not. From this, one gets

$$\mathbf{\Gamma}_t = \frac{c^2 f'}{4}\gamma^{rt} = \frac{cf'}{4(fh)^{\frac{1}{2}}}\hat{\gamma}^{10}, \tag{6.9}$$

$$\mathbf{\Gamma}_r = 0, \tag{6.10}$$

$$\mathbf{\Gamma}_\theta = -\frac{r}{2}\gamma^{r\theta} = -\frac{1}{2h^{\frac{1}{2}}}\hat{\gamma}^{12}, \tag{6.11}$$

$$\begin{aligned}\mathbf{\Gamma}_\phi &= -\frac{r\sin^2\theta}{2}\gamma^{r\phi} - \frac{r^2\sin\theta\cos\theta}{2}\gamma^{\theta\phi}\\ &= -\frac{\sin\theta}{2h^{\frac{1}{2}}}\hat{\gamma}^{13} - \frac{\cos\theta}{2}\hat{\gamma}^{23}.\end{aligned} \tag{6.12}$$

To obtain the curvature 2-forms we use Eq. (5.72), that is

$$\tfrac{1}{2}\mathscr{R}_{\alpha\beta} = \frac{\partial}{\partial u^\alpha}\mathbf{\Gamma}_\beta - \frac{\partial}{\partial u^\beta}\mathbf{\Gamma}_\alpha + \mathbf{\Gamma}_\beta\mathbf{\Gamma}_\alpha - \mathbf{\Gamma}_\alpha\mathbf{\Gamma}_\beta.$$

From this formula, we get

$$\begin{aligned}\tfrac{1}{2}\mathscr{R}_{rt} = \frac{\partial}{\partial r}\mathbf{\Gamma}_t &= \left[\frac{cf''}{4(fh)^{\frac{1}{2}}} - \frac{cf'(fh)'}{8(fh)^{3/2}}\right]\hat{\gamma}^{10}\\ &= -\left[\frac{f''}{4(fh)} - \frac{f'(fh)'}{8(fh)^2}\right]\gamma_{rt}.\end{aligned} \tag{6.13}$$

In a similar fashion

$$\tfrac{1}{2}\mathscr{R}_{\theta t} = -\frac{f'}{4(fh)r}\gamma_{\theta t}, \tag{6.14}$$

$$\tfrac{1}{2}\mathscr{R}_{\phi t} = -\frac{f'}{4(fh)r}\gamma_{\phi t}, \tag{6.15}$$

$$\tfrac{1}{2}\mathscr{R}_{r\theta} = \frac{h'}{4(h)^2 r}\gamma_{r\theta}, \tag{6.16}$$

$$\tfrac{1}{2}\mathscr{R}_{\phi r} = \frac{h'}{4(h)^2 r}\gamma_{\phi r}, \tag{6.17}$$

$$\tfrac{1}{2}\mathscr{R}_{\theta\phi} = \frac{h^{-1}}{2hr^2}\gamma_{\theta\phi}. \tag{6.18}$$

From Problem 5.14 and Eq. (5.66), Einstein's field equations can be written in the form

$$\gamma^{\beta}\mathscr{R}_{\beta\alpha} = 2R_{\alpha p}\gamma^{p} = 0. \tag{6.19}$$

Thus we have

$$\begin{aligned}\gamma^{\beta}\mathscr{R}_{\beta t} &= \gamma^{r}\mathscr{R}_{rt} + \gamma^{\theta}\mathscr{R}_{\theta t} + \gamma^{\beta}\mathscr{R}_{\phi t}\\ &= -\left[\frac{f''}{2(fh)} - \frac{f'(fh)'}{4(fh)^2}\right]\gamma_t - \frac{f'}{2(fh)r}\gamma_t - \frac{f'}{2(fh)r}\gamma_t\\ &= 0.\end{aligned}$$

or

$$\gamma^{\beta}\mathscr{R}_{\beta t} = -\left[\frac{f''}{2(fh)} - \frac{f'(fh)'}{4(fh)^2} + \frac{f'}{(fh)r}\right]\gamma_t = 0. \tag{6.20}$$

In a similar fashion:

$$\gamma^{\beta}\mathscr{R}_{\beta r} = -\left[\frac{f''}{2(fh)} - \frac{f'(fh)'}{4(fh)^2} - \frac{h'}{(h)^2 r}\right]\gamma_r = 0. \tag{6.21}$$

$$\gamma^{\beta}\mathscr{R}_{\beta\theta} = \left[\frac{-f'}{2(fh)r} + \frac{h'}{2(h)^2 r} + \frac{h-1}{hr^2}\right]\gamma_{\theta} = 0, \tag{6.22}$$

and

$$\gamma^{\beta}\mathscr{R}_{\beta\phi} = \left[\frac{-f'}{2(fh)r} + \frac{h'}{2(h)^2 r} + \frac{h-1}{hr^2}\right]\gamma_{\phi} = 0. \tag{6.23}$$

Subtracting Eq. (6.20) from Eq. (6.21) gives us

$$\frac{h'}{(h)^2 r} + \frac{f'}{(fh)r} = 0 \tag{6.24}$$

or

$$\frac{h'}{h} + \frac{f'}{f} = 0.$$

Integrating, we get

$$\ln(h) + \ln(f) = \ln(fh) = \ln(k)$$

or $f(r)h(r) = k$. At long distances from the sun, our metric should approach the flat space-time Lorentz metric, so

$$\lim_{r \to \infty} f(r) = \lim_{r \to \infty} h(r) = 1.$$

Thus it is appropriate to set our constant k equal to 1 and we then have

$$f(r)h(r) = 1. \tag{6.25}$$

From Eqs. (6.24) and (6.25), Eq. (6.22) becomes

$$\frac{h'}{(h)^2 r} + \frac{h-1}{hr^2} = 0 \qquad \text{or} \qquad \frac{h'}{h(h-1)} + \frac{1}{r} = 0.$$

From the partial fraction expansion, this becomes

$$-\frac{h'}{h} + \frac{h'}{h-1} + \frac{1}{r} = 0.$$

Integrating this equation gives us

$$-\ln(h) + \ln(h-1) + \ln(r) = \ln C$$

or

$$\frac{r(h-1)}{h} = C.$$

Solving for h, we get

$$h(r) = \left(1 - \frac{C}{r}\right)^{-1}. \tag{6.26}$$

From Eq. (6.25), we also have

$$f(r) = \left(1 - \frac{C}{r}\right). \tag{6.27}$$

The reader should also check that these formulas for f and h also satisfy Eqs. (6.20) and (6.21). Traditionally the constant C is designated by $2m$, so the line element for the Schwarzschild metric becomes

$$(ds)^2 = \left(1 - \frac{2m}{r}\right)c^2(dt)^2 - \left(1 - \frac{2m}{r}\right)^{-1}(dr)^2 - r^2[(d\theta)^2 + r^2 \sin^2 \theta(d\phi)^2]. \tag{6.28}$$

In the next section, it will become evident that the constant

$$m = \frac{MG}{c^2}$$

where M is the mass of the sun and G is the universal gravitational constant.

A singularity of sorts occurs in the metric at $r = 2m$. For this reason, $2m$ is known as the *Schwarzschild radius*. In the study of black holes, one must consider the nature of physical phenomena for $r \leqslant 2m$. However, for the study of planetary motion this is not a matter of concern. The field equations used to derive the Schwarzschild metric are valid only in a mass free space. For the sun, the Schwarzschild radius is approximately 3 km, which is well within the surface of the sun which is the boundary of the mass free region.

In this section, we constructed curvature 2-forms from Fock–Ivanenko 2-vectors and then used the resulting curvature 2-forms to obtain Einstein's field equations. Conventional wisdom says that the most efficient method of obtaining curvature 2-forms for a not too complex metric is to first compute what are known as connection 1-forms via a "guess and check" method introduced by Professor C. W. Misner (1963, appendix A of paper). This method is used to compute the Schwarzschild metric in Wald's *General Relativity* (1984, pp. 51–52 and pp. 121–127). A similar calculation is carried out for the Friedman metric in *Gravitation* (Misner, Thorne, and Wheeler 1973, pp. 355–357). It is my opinion that the method used in this section is both quicker and more straightforward.

In the next section, we will compute the bounded orbit of a planet predicted by the Schwarzschild metric and compare the result with the classical Newtonian computation.

Problem 6.1. Even in classical Newtonian physics, the Schwarzschild radius has an interesting significance. Suppose the mass of a spherical astronomical body (star, planet, or whatever) is so dense that the radius of the body is equal to $2MG/c^2$. Using the assumptions of Newtonian physics, show that the escape velocity for an object initially on the surface of such a body is equal to the speed of light.

6.2 The Precession of Perihelion for Mercury

This section contains no applications of Clifford algebra. This section is included to give credence to the claim that there is some significance to the Schwarzschild metric that was computed in the last section. Readers who have a cursory knowledge of general relativity may wish to skip this section.

For those who insist on not skipping this section, we will compare the equations of motion for a planet according to the general theory of relativity with the equations of motion according to Newtonian physics.

In classical analytic mechanics, the path of a particle is that of the "least action" which is determined by the calculus of variation equation:

$$\delta \int \left[\tfrac{1}{2} m g_{\alpha\beta} \frac{du^\alpha}{dt} \frac{du^\beta}{dt} - V(u^1, u^2, u^3) \right] dt = 0.$$

In the case of planetary motion in spherical coordinates, we have

$$\delta \int \left(\tfrac{1}{2} m \left[\left(\frac{dr}{dt} \right)^2 + \left(r \frac{d\theta}{dt} \right)^2 + \left(r \sin\theta \frac{d\phi}{dt} \right)^2 \right] + \frac{mMG}{r} \right) dt = 0. \tag{6.29}$$

To cast this classical equation in a form which will make it comparable with that of Einstein, it is useful to replace t by $s = ct$. If we also factor out a factor of $\frac{1}{2}mc^2$, Eq. (6.29) becomes

$$\delta \int \left[(\dot{r})^2 + (r\dot{\theta})^2 + (r \sin\theta \dot{\phi})^2 + \frac{2MG}{c^2 r} \right] ds = 0, \tag{6.30}$$

where it should be understood that the dot above the r, θ, and ϕ indicates a differentiation with respect to s.

The Euler–Lagrange equations for Eq. (6.30) are

$$\frac{d}{ds} \frac{\partial L}{\partial \dot{u}^\alpha} = \frac{\partial L}{\partial u^\alpha}$$

where L designates the integrand of Eq. (6.30) and u^1, u^2, u^3 represent respectively r, θ, and ϕ.

The Euler–Lagrange equations for θ and ϕ are

$$\frac{d}{ds}(2r^2\dot{\theta}) = 2r^2 \sin\theta \cos\theta (\dot{\phi})^2 \tag{6.31}$$

and

$$\frac{d}{ds}[2r^2 \sin^2\theta (\dot{\phi})] = 0. \tag{6.32}$$

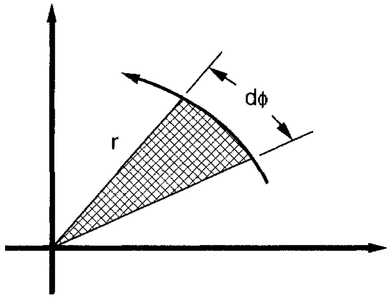


Fig. 6.1. The rate at which the shaded area is swept out by the radius vector is the *areal velocity*. The fact that the areal velocity is constant for planetary motion about the sun is *Kepler's second law*.

It is well known that a particle acted on by a central force stays in a plane. We can choose this plane to be $\theta = \pi/2$. This solves Eq. (6.31). Equation (6.32) then becomes

$$\frac{\mathrm{d}}{\mathrm{d}s}(r^2\dot{\phi}) = 0$$

or

$$\tfrac{1}{2}r^2\dot{\phi} = \alpha_0/c. \tag{6.33}$$

The constant α_0 is chosen so that it has a standard interpretation. In particular

$$\alpha_0 = \tfrac{1}{2}r^2\frac{\mathrm{d}\phi}{\mathrm{d}t}.$$

To interpret this, we note that $\frac{1}{2}r^2\,\mathrm{d}\phi$ represents an infinitesimal area swept out by the radius vector from the sun to the planet. (See Fig. 6.1.) Thus α_0 is the *areal velocity*—that is the rate at which this area is swept out. The fact that the areal velocity is constant is *Kepler's second law.*

Rather than examine the Euler–Lagrange equation for r, it is easier to take advantage of the fact that s does not appear explicitly in the integrand of Eq. (6.30). This means that

$$\left(\dot{u}^\alpha\frac{\partial L}{\partial\dot{u}^\alpha} - L\right)\text{ is constant,}$$

and this gives us our constant energy equation:

$$(\dot{r})^2 + 4(\alpha_0)^2/(c^2r^2) - 2MG/(c^2r) = 2E/(mc^2)$$

or

$$(\dot{r})^2 = 2E/(mc^2) + 2MG/(c^2 r) - 4(\alpha_0)^2/(c^2 r^2), \tag{6.34}$$

where E is a negative constant representing the total Newtonian mechanical energy of the system and m is the mass of the planet.

Equation (6.34) is more tractable if we substitute

$$r = \frac{1}{u}. \tag{6.35}$$

Furthermore, the resulting equation is easier to solve for u as a function of ϕ rather than as a function of s or t. From Eq. (6.35)

$$\dot{r} = -\frac{1}{u^2}\frac{du}{d\phi}\dot{\phi}. \tag{6.36}$$

From Eq. (6.33),

$$\dot{\phi} = 2\alpha_0/(cr^2) = 2\alpha_0(u)^2/c. \tag{6.37}$$

Using Eqs. (6.35), (6.36), and (6.37), Eq. (6.34) becomes

$$\left(\frac{du}{d\phi}\right)^2 = \frac{E}{2m\alpha_0^2} + \frac{MG}{2\alpha_0^2}u - u^2 = P_C(u). \tag{6.38}$$

The polynomial $P_C(u)$ is of course a parabola. Since the left-hand side of Eq. (6.38) must be positive, the physically meaningful values of u are those for which $P_C(u)$ is positive. (See Fig. 6.2.)

If we designate the two roots of the polynomial by u_{MAX} and u_{MIN} then

$$P_C(u) = (u_{\text{MAX}} - u)(u - u_{\text{MIN}}). \tag{6.39}$$

Using this form for $P_C(u)$, we can rewrite Eq. (6.38) to obtain

$$d\phi = \frac{\pm du}{\sqrt{(u_{\text{MAX}} - u)(u - u_{\text{MIN}})}}. \tag{6.40}$$

Suppose the planet moves in the direction of increasing ϕ and suppose that we start the measurement of ϕ from the position of perihelion where $r = r_{\text{MIN}}$ and $u = u_{\text{MAX}}$. From that position, u decreases while ϕ increases. Thus we must choose the negative sign in Eq. (6.40) so

$$\phi = -\int_{u_{\text{MAX}}}^{u} \frac{dx}{\sqrt{(u_{\text{MAX}} - x)(x - u_{\text{MIN}})}}. \tag{6.41}$$

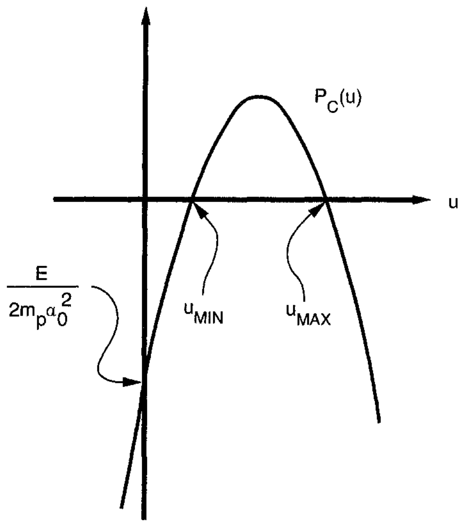


Fig. 6.2. For physically meaningful values of u, the parabola $P_C(u) = (u_{MAX} - u)(u - u_{MIN})$ must be non-negative.

By completing the square, the integrand can be reformulated so that

$$\phi = -\int_{u_{MAX}}^{u} \frac{dx}{\sqrt{(c/b^2)^2 - (x - a/b^2)^2}}, \tag{6.42}$$

where

$$\frac{c}{b^2} = \frac{u_{MAX} - u_{MIN}}{2}$$

and

$$\frac{a}{b^2} = \frac{u_{MAX} + u_{MIN}}{2}.$$

Carrying out the integration, we have

$$\phi = \arccos\left[\left(x - \frac{a}{b^2}\right)\bigg/\left(\frac{c}{b^2}\right)\right]\Bigg|_{u_{MAX}=(a+c)/b_2}^{u}$$

and thus

$$\phi = \arccos\left[\left(u - \frac{a}{b^2}\right)\bigg/\left(\frac{c}{b^2}\right)\right] \tag{6.43}$$

or

$$u = \frac{1}{r} = \frac{a}{b^2} + \frac{c}{b^2}\cos\phi. \tag{6.44}$$

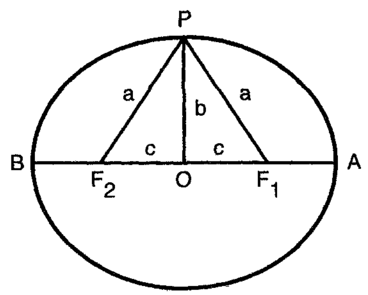


Fig. 6.3. The ellipse $x^2/a^2 + y^2/b^2 = 1$ with focal points at F_1 and F_2.

This is the equation for an ellipse where r is the distance from one of the focal points, a is the magnitude of the semi-major axis, b is the magnitude of the semi-minor axis, c is half the distance between the focal points, and ϕ is the angle swept out by the radius vector from the position of perihelion. (See Figs. 6.3 and 6.4 and Problem 6.2.)

The fact that, according to Newtonian physics, the orbit of a planet is represented by an ellipse with the sun at one of the focal points is *Kepler's first law.*

Equation (6.43) is valid only between the initial perihelion point and the first aphelion point encountered by the planet. At aphelion $r = r_{\mathrm{MAX}}$ and $u = u_{\mathrm{MIN}} = (a - c)/b^2$, so at that point

$$\phi = \operatorname{arc\,cos}\left[\left(\frac{-c}{b^2}\right)\Big/\left(\frac{c}{b^2}\right)\right] = \operatorname{arc\,cos}(-1) = \pi.$$

Immediately after the point of aphelion, both du and dϕ are positive. The equation for ϕ then becomes

$$\phi = \pi + \int_{u_{\mathrm{MIN}}}^{u} \frac{\mathrm{d}x}{\sqrt{(c/b^2)^2 - (x - a/b^2)^2}}$$

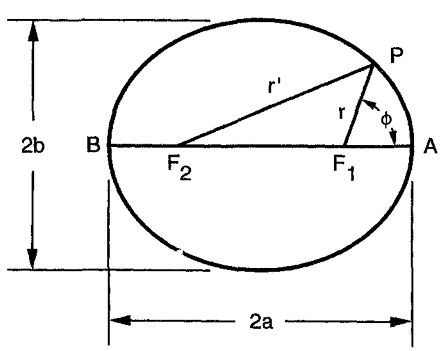


Fig. 6.4. The ellipse $1/r = a/b^2 + (c/b^2)\cos\phi$. This is the same ellipse as shown in Fig. 6.3.

or

$$\phi = \pi - \text{arc cos}\left[\left(u - \frac{a}{b^2}\right)\Big/\left(\frac{c}{b^2}\right)\right] + \text{arc cos}(-1).$$

Thus

$$\phi = 2\pi - \text{arc cos}\left[\left(u - \frac{a}{b^2}\right)\Big/\left(\frac{c}{b^2}\right)\right]$$

or

$$u = \frac{1}{r} = \frac{a}{b^2} + \frac{c}{b^2}\cos(2\pi - \phi) = \frac{a}{b^2} + \frac{c}{b^2}\cos\phi.$$

Therefore Eq. (6.44) remains valid for the entire orbit even though Eq. (6.43) is valid for only one half the orbit.

It is also possible to get a simple formula for the period of revolution τ. The formula is

$$\tau^2 = (4\pi^2 a^3)/(MG). \tag{6.45}$$

To derive Eq. (6.45), we first note that α_0 represents the areal velocity so that

$$\alpha_0 = \frac{\mathrm{d}A}{\mathrm{d}t}$$

where A is the area swept out by the radius vector from the sun to the planet. Integrating over one period of revolution, we get the area of the ellipse, that is

$$A = \pi ab = \alpha_0 \tau. \tag{6.46}$$

To get our desired result from Eq. (6.46), we need a formula for α_0. This can be obtained by combining Eqs. (6.38) and (6.39). This gives us

$$(u_{\mathrm{MAX}} - u)(u - u_{\mathrm{MIN}}) = \frac{E}{2m\alpha_0^2} + \frac{MG}{2\alpha_0^2}u - u^2.$$

Equating the constant coefficients of u, we have

$$u_{\mathrm{MAX}} + u_{\mathrm{MIN}} = \frac{MG}{2\alpha_0^2}.$$

But from Problem 6.2, $u_{\mathrm{MAX}} + u_{\mathrm{MIN}} = 2a/b^2$. Thus

$$\alpha_0^2 = \frac{MG}{2(u_{\mathrm{MAX}} + u_{\mathrm{MIN}})} = \frac{MGb^2}{4a}. \tag{6.47}$$

Thus, from Eqs. (6.46) and (6.47)

$$\tau^2 = \frac{\pi^2 a^2 b^2}{\alpha_0^2} = \frac{4\pi^2 a^3}{MG}$$

and this is our desired result.

The relation that the square of the period of each planet is proportional to the cube of the semi-major axis is *Kepler's third law.*

The fact that Newton could derive Kepler's three laws from the simple inverse square force law was a great triumph for Newton's theory of gravity. Einstein's theory of general relativity adds only a very minuscule correction to these results.

We now turn to the problem of determining the planetary orbits predicted by Einstein's theory. In Einstein's theory, the planetary orbits are geodesics of the Schwarzschild metric. Perhaps the quickest way to obtain the equations for the geodesics is to get the Euler–Lagrange equations for the calculus of variation problem

$$\delta \int \left[\left(1 - \frac{2m}{r}\right) c^2(\dot{t})^2 - \left(1 - \frac{2m}{r}\right)^{-1} (\dot{r})^2 - r^2(\dot{\theta})^2 - r^2 \sin^2\theta(\dot{\phi})^2 \right] \mathrm{d}s = 0. \tag{6.48}$$

(See Prob. 6.3.)

As in Eq. (6.30), the dot above the t, r, and ϕ in Eq. (6.48) indicates differentiation with respect to s. Three of the four Euler–Lagrange equations are as follows:

$$\frac{\mathrm{d}}{\mathrm{d}s}\left[\left(1 - \frac{2m}{r}\right)\dot{t}\right] = 0 \tag{6.49}$$

$$\frac{\mathrm{d}}{\mathrm{d}s}(r^2\dot{\theta}) = r^2 \sin\theta \cos\theta(\dot{\phi})^2, \tag{6.50}$$

$$\frac{\mathrm{d}}{\mathrm{d}s}(r^2 \sin^2\theta\dot{\phi}) = 0. \tag{6.51}$$

We note that Eqs. (6.50) and (6.51) are identical to their classical counterparts. In solving the Euler–Lagrange equations, we again pick out the plane for which $\theta = \pi/2$. We then have, as before:

$$\frac{\mathrm{d}}{\mathrm{d}s}(r^2\dot{\phi}) = 0 \qquad \text{and} \qquad \tfrac{1}{2}r^2\dot{\phi} = \alpha_0/c. \tag{6.52}$$

Furthermore, one can integrate Eq. (6.49) to get

$$\left(1-\frac{2m}{r}\right)\dot{t}=k, \tag{6.53}$$

where k is understood to be a constant.

In solving the Newtonian system of equations, we got Eq. (6.34) by taking advantage of the fact that s did not appear explicitly in the integrand of Eq. (6.30). We could do the same thing here, but it is easier to obtain the same result by simply noting that from Eq. (6.28),

$$\left(\frac{\mathrm{d}s}{\mathrm{d}s}\right)^2=1=\left(1-\frac{2m}{r}\right)c^2(\dot{t})^2-\left(1-\frac{2m}{r}\right)^{-1}(\dot{r})^2-r^2(\dot{\theta})^2-r^2\sin\theta(\dot{\phi})^2. \tag{6.54}$$

This simplifies when we set $\theta=\pi/2$ and $\dot{\theta}=0$. In addition, Eqs. (6.52) and (6.53) can be used to eliminate $\dot{\phi}$ and $\dot{t}$ from Eq. (6.54). The result is

$$(\dot{r})^2=-(1-k^2c^2)+\frac{2m}{r}-\frac{4\alpha_0^2}{c^2r^2}+\frac{8m\alpha_0^2}{c^2r^3}. \tag{6.55}$$

Comparing Eq. (6.55) with its Newtonian counterpart Eq. (6.34), we see that if we identify m with MG/c^2, the two equations are virtually identical except for the last term on the right-hand side of Eq. (6.55).

As before, we substitute $r=1/u$ and change the independent variable from s to ϕ. Following the previous calculation, we have

$$(\dot{r}=-\left(\frac{1}{u^2}\right)\dot{u}=-\left(\frac{1}{u^2}\right)\frac{\mathrm{d}u}{\mathrm{d}\phi}\dot{\phi}=-2\left(\frac{\alpha_0}{c}\right)\frac{\mathrm{d}u}{\mathrm{d}\phi}$$

and Eq. (6.55) becomes

$$\begin{aligned}\left(\frac{\mathrm{d}u}{\mathrm{d}\phi}\right)^2&=-\frac{(1-k^2c^2)c^2}{4(\alpha_0)^2}+\frac{mc^2}{2(\alpha_0)^2}u-(u)^2+2m(u)^3\\&=P_R(u).\end{aligned} \tag{6.56}$$

For a planetary orbit, $P_R(u)$ must not only be nonnegative but u must range between two roots of $P_R(u)$ which are labeled by u_{MIN} and u_{MAX} in Fig. 6.5.

Writing $P_R(u)$ in terms of these two roots give us

$$P_R(u)=(u_{\mathrm{MAX}}-u)(u-u_{\mathrm{MIN}})(A+Bu). \tag{6.57}$$

Multiplying out the right-hand side of Eq. (6.57) and comparing the

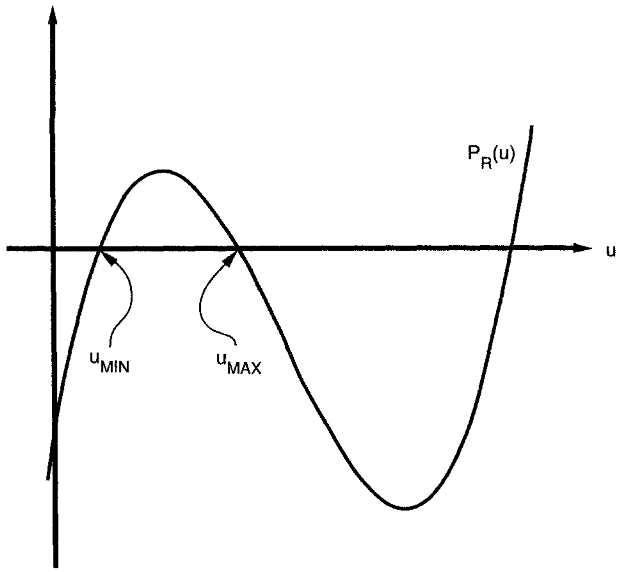


Fig. 6.5. The cubic polynomial $P_R(u) = (u_{MAX} - u)(u - u_{MIN})(A + Bu)$ must be nonnegative for physically meaningful values of u. For a planetary orbit, u must range between u_{MIN} and u_{MAX}.

coefficients of $(u)^2$ and $(u)^3$ with those in Eq. (6.56), it becomes clear that

$$B = -2m \tag{6.58}$$

and

$$A = 1 - 2m(u_{MAX} + u_{MIN}) = 1 - \frac{4ma}{b^2}. \tag{6.59}$$

Equations (6.57), (6.58), and (6.59) can now be used to modify Eq. (6.56). If ϕ is measured as before from the position of perihelion, then Eq. (6.56) rewritten in integral form becomes

$$\phi = -\int_{u_{MAX}}^{u} \frac{\mathrm{d}x}{\sqrt{(u_{MAX} - x)(x - u_{MIN})(1 - (4ma/b^2) - 2mx}}. \tag{6.60}$$

Expanding the integrand of Eq. (6.60) as a power series in m and retaining only first-order terms results in the relation,

$$\phi = -\int_{u_{MAX}}^{u} \frac{(1 + (2ma/b^2) + mx)\,\mathrm{d}x}{\sqrt{(u_{MAX} - x)(x - u_{MIN})}}. \tag{6.61}$$

With this approximation, it is possible to carry out the integration in much the same way as the corresponding integral was done for the

Newtonian orbit. By completing the square for the quadratic under the square root sign and regrouping the terms in the numerator, Eq. (6.61) becomes

$$\phi = -\int_{u_{\mathrm{MAX}}}^{u} \frac{[(1 + 3ma/b^2) + m(x - a/b^2)]\,dx}{\sqrt{(c/b^2)^2 - (x - a/b^2)^2}}. \tag{6.62}$$

Carrying out the integration, we have

$$\phi = \left(1 + \frac{3ma}{b^2}\right) \operatorname{arc\,cos}\left(\frac{u - a/b^2}{c/b^2}\right) + [(u_{\mathrm{MAX}} - u)(u - u_{\mathrm{MIN}})]^{\frac{1}{2}}. \tag{6.63}$$

At the first aphelion, $u = u_{\mathrm{MIN}} = (a - c)/b^2$ and

$$\phi = \left(1 + \frac{3ma}{b^2}\right) \operatorname{arc\,cos}(-1) = \pi\left(1 + \frac{3ma}{b^2}\right).$$

When the planet attains the position of perihelion again

$$\phi = 2\pi\left(1 + \frac{3ma}{b^2}\right).$$

Therefore over each period there is an advance of perihelion by the amount

$$\Delta\phi = 6\pi \frac{ma}{b^2}.$$

Noting that $m = MG/c^2$ and $\tau^2 = 4\pi^2 a^3/MG$, we have

$$\frac{\Delta\phi}{\tau} = \frac{3(MG)^{3/2}}{c^2} \frac{1}{a^{\frac{1}{2}} b^2}. \tag{6.64}$$

For the case of Mercury, Eq. (6.64) translates into 43.03″ per century. Actually the gravitational forces due to other planets cause Mercury to precess at a rate far greater than this. However, during the middle of the nineteenth century, it became clear that the gravitational forces due to the known planets could not completely account for the precession of Mercury.

At the time, the natural explanation was that there existed a still undiscovered planet between Mercury and the sun. Anomalous behavior in the orbit of Uranus had also been observed. The British astronomer John Couch Adams and the French astronomer Urbain Jean Joseph Leverrier independently made very lengthy computations to predict the position of the then undiscovered planet Neptune. Leverrier sent his results to the German astronomer Johann Gottfried Galle of the Berlin Observatory. With this information, Galle was able to locate Neptune on September 23, 1846.

When Leverrier turned his attention to the advance in the periheliion of Mercury, he found that he could account for the unexplained portion of the advance by hypothesizing the existence of a planet between Mercury and the sun. This theoretical planet became known as Vulcan and many attempts were made to find it. These attempts all ended in failure. In 1898, the American astronomer Simon Newcomb published a paper in which he calculated the residual advance unaccounted by the known planets to be $41.24'' \pm 2.09''$ per century (Newcomb 1898). More accurate estimates of this residual advance have been made in recent years, but they remain consistent with the value of $43.03''$ predicted by Einstein's theory (1915c).

Problem 6.2. A point on an ellipse can be characterized by the fact that its distance to one focal point plus its distance to the other focal point is equal to some constant which is the magnitude of the major axis. Referring to Figs. 6.3 and 6.4, it is clear from the law of cosines that

$$(r')^2 = r^2 + (2c)^2 + 4cr\cos\phi.$$

(1) Using the fact that $r' + r = 2a$, show that

$$\frac{1}{r} = \frac{a}{b^2} + \frac{c}{b^2}\cos\phi.$$

(From Fig. 6.3, it should be clear that $b^2 = a^2 - c^2$.)

(2) From Figs. 6.3 and 6.4, it follows that $r_{\text{MIN}} = a - c$ and $r_{\text{MAX}} = a + c$. Use this to show that

$$\frac{u_{\text{MAX}} + u_{\text{MIN}}}{2} = \frac{a}{b^2} \quad \text{and} \quad \frac{u_{\text{MAX}} - u_{\text{MIN}}}{2} = \frac{c}{b^2}$$

where $u = 1/r$ and $u_{\text{MAX}} = 1/r_{\text{MIN}}$ while $u_{\text{MIN}} = 1/r_{\text{MAX}}$.

Problem 6.3. Suppose

$$I = \int_A^B \mathrm{d}s = \int_A^B L\,\mathrm{d}\lambda$$

where $L = (g_{\alpha\beta}\dot{u}^\alpha\dot{u}^\beta)^{\frac{1}{2}}$ and $\dot{u}^\alpha = \mathrm{d}u^\alpha/\mathrm{d}\lambda$.

The problem of determining the equation for a geodesic path is equivalent to solving the calculus of variations problem $\delta I = 0$.

(1) Show that the Euler–Lagrange equations for this problem can be put in the form

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(L^{-1}\frac{\mathrm{d}u^\alpha}{\mathrm{d}\lambda}\right) + L^{-1}\Gamma_{\eta\nu}{}^{\eta}\frac{\mathrm{d}u^\eta}{\mathrm{d}\lambda}\frac{\mathrm{d}u^\nu}{\mathrm{d}\lambda} = 0. \tag{6.65}$$

(2) Using the fact that

$$\mathrm{d}s = L\,\mathrm{d}\lambda \qquad \text{or} \qquad L^{-1}\frac{\mathrm{d}u^\alpha}{\mathrm{d}\lambda} = \frac{\mathrm{d}\lambda}{\mathrm{d}s}\frac{\mathrm{d}u^\alpha}{\mathrm{d}\lambda} = \frac{\mathrm{d}u^\alpha}{\mathrm{d}s},$$

show that

$$\frac{\mathrm{d}^2u^\alpha}{\mathrm{d}s^2} + \Gamma_{\eta\nu}{}^\alpha \frac{\mathrm{d}u^\eta}{\mathrm{d}s}\frac{\mathrm{d}u^\nu}{\mathrm{d}s} = 0. \tag{6.66}$$

(3) Suppose $I = \int_A^B [L^2]\,\mathrm{d}s$. Consider the problem of solving the equation $\delta I = 0$. Show that the Euler–Lagrange equations for this problem also result in the system of equations for a geodesic. (Eq. (6.66.)

7

TWO DIFFERENTIAL OPERATORS

7.1 The Exterior Derivative d and the Codifferential Operator δ Related to the Operator $\nabla = \gamma^j \nabla_j$

The operator that appears in Dirac's equation for the electron is $\nabla = \gamma^j \nabla_j$. Let us examine what happens when this operator is applied to a p-vector in a space that may be curved. Suppose our p-vector is written in the form

$$\mathscr{F} = \frac{1}{p!} F_{j_1 j_2 \ldots j_p} \gamma^{j_1 j_2 \ldots j_p}, \tag{7.1}$$

where the components of $\mathscr{F}$ are assumed to be totally antisymmetric with respect to their indices. Then using a modified form of Eq. (5.20), we have

$$\nabla \mathscr{F} = \gamma^k \nabla_k \mathscr{F} = \frac{1}{p!} F_{j_1 j_2 \ldots j_p ; k} \gamma^k \gamma^{j_1 j_2 \ldots j_p}. \tag{7.2}$$

Now let us consider the product $\gamma^k \gamma^{j_1 j_2 \ldots j_p}$. To get a handle on this product, suppose we were dealing with an orthonormal system of Dirac matrices. We would then have

$$\hat{\gamma}^k \hat{\gamma}^{j_1 j_2 \ldots j_p} = \hat{\gamma}^k \hat{\gamma}^{j_1} \hat{\gamma}^{j_2} \ldots \hat{\gamma}^{j_p}.$$

If the index k is distinct from all the j_r's then this product is the $(p+1)$-vector

$$\hat{\gamma}^{k j_1 j_2 \ldots j_p}.$$

On the other hand, suppose $k = j_r$. In that case, the product becomes a $(p-1)$-vector. In particular

$$\begin{aligned} \hat{\gamma}^k \hat{\gamma}^{j_1} \hat{\gamma}^{j_2} \ldots \hat{\gamma}^{j_p} &= (-1)^{r-1} \hat{\gamma}^k \hat{\gamma}^{j_r} \hat{\gamma}^{j_1} \hat{\gamma}^{j_2} \ldots \widehat{\hat{\gamma}^{j_r}} \ldots \hat{\gamma}^{j_p} \\ &= (-1)^{r-1} \eta^{k j_r} \hat{\gamma}^{k_1} \hat{\gamma}^{j_2} \ldots \widehat{\hat{\gamma}^{j_p}} \ldots \hat{\gamma}^{j_p}, \end{aligned}$$

where $\widehat{\hat{\gamma}^{j_r}}$ denotes the fact that $\hat{\gamma}^{j_r}$ is missing from its original position in the product. Summarizing the possibilities, one has

$$\hat{\gamma}^k\hat{\gamma}^{j_1j_2\ldots j_p} = \hat{\gamma}^{kj_1j_2\ldots j_p} + \sum_{r=1}^{p} (-1)^{r-1}\eta^{kj_r}\hat{\gamma}^{j_1j_2\ldots\hat{j}_r\ldots j_p}, \tag{7.3}$$

where $\hat{j}_r$ denotes a missing index.

Equation (7.3) easily generalizes to any system of Dirac matrices. (See Problem 7.1.) In general one has

$$\gamma^k\gamma^{j_1j_2\ldots j_p} = \gamma^{kj_1j_2\ldots j_p} + \sum_{r=1}^{p} (-1)^{r-1}g^{kj_r}\gamma^{j_1j_2\ldots\hat{j}_r\ldots j_p}. \tag{7.4}$$

Inserting this result into Eq. (7.2) gives us

$$\begin{aligned}\nabla\mathscr{F} &= \frac{1}{p!}F_{j_1j_2\ldots j_p;k}\gamma^{j_1j_2\ldots j_p}\\ &\quad+\frac{1}{p!}\sum_{r=1}^{p}(-1)^{r-1}F_{j_1j_2\ldots j_p,k}g^{kj_r}\gamma^{j_1j_2\ldots\hat{j}_r\ldots j_p}.\end{aligned} \tag{7.5}$$

But

$$\begin{aligned}F_{j_1j_2\ldots j_p;k}g^{kj_r} &= (-1)^{r-1}F_{j_rj_1j_2\ldots\hat{j}_r\ldots j_p;k}g^{kj_r}\\ &= (-1)^{r-1}F^k{}_{j_1j_2\ldots\hat{j}_r\ldots j_p;k}.\end{aligned}$$

Thus

$$\begin{aligned}\nabla\mathscr{F} &= \frac{1}{p!}F_{j_1j_2\ldots j_p;k}\gamma^{kj_1j_2\ldots j_p}\\ &\quad+\frac{1}{p!}\sum_{r=1}^{p}F^k{}_{j_1j_2\ldots\hat{j}_r\ldots j_p;k}\gamma^{j_1j_2\ldots\hat{j}_r\ldots j_p}.\end{aligned} \tag{7.6}$$

If we relabel the index sequence $j_1j_2\ldots\hat{j}_r\ldots j_p$, by $j_1j_2\ldots j_{p-1}$, it becomes clear that the sum on the right-hand side of Eq. (7.6) consists of p identical terms. Thus we finally have

$$\begin{aligned}\nabla\mathscr{F} &= \frac{1}{p!}F_{j_1j_2\ldots j_p,k}\gamma^{kj_1j_2\ldots j_p}\\ &\quad+\frac{1}{(p-1)!}F^k{}_{j_1j_2\ldots j_{p-1};k}\gamma^{j_1j_2\ldots j_{p-1}}.\end{aligned} \tag{7.7}$$

We now split $\nabla\mathscr{F}$ into two pieces—a $(p+1)$-vector and a $(p-1)$-vector. This decomposition now enables us to define the *exterior derivative* $\mathbf{d}$ and

the *codifferential operator* $\boldsymbol{\delta}$. In particular

$$\mathbf{d}\mathscr{F} = \frac{1}{p!} F_{j_1 j_2 \dots j_p;k} \gamma^{k j_1 j_2 \dots j_p} \tag{7.8}$$

and

$$\begin{aligned} \boldsymbol{\delta}\mathscr{F} &= \frac{1}{(p-1)!} F^k{}_{j_1 j_2 \dots j_{p-1};k} \gamma^{j_1 j_2 \dots j_{p-1}} \\ &= \frac{1}{(p-1)!} F^{k j_1 j_2 \dots j_{p-1}}{}_{;k} \gamma_{j_1 j_2 \dots j_{p-1}}, \end{aligned} \tag{7.9}$$

where in both cases

$$\mathscr{F} = \frac{1}{p!} F_{j_1 j_2 \dots j_p} \gamma^{j_1 j_2 \dots j_p}.$$

It should be noted that the sign for the exterior derivative $\mathbf{d}$ is well established and we have followed that convention. On the other hand, the sign convention for the codifferential operator $\boldsymbol{\delta}$ is not so well established. For the case of a positive definite metric, our sign convention is identical to that used by Heinrich W. Guggenheimer (1977, p. 329). It is opposite to the sign used by Harley Flanders (1963, pp. 136–137).

In coordinate frames, both Eq. (7.8) and (7.9) can be cast in more useful forms. Let us first consider Eq. (7.8):

$$\begin{aligned} \mathbf{d}\mathscr{F} &= \frac{1}{p!} F_{\beta_1 \beta_2 \dots \beta_p, \alpha} \gamma^{\alpha \beta_1 \beta_2 \dots \beta_p} \\ &= \frac{1}{p!} \left(\frac{\partial}{\partial u^\alpha} F_{\beta_1 \beta_2 \dots \beta_p} - \sum_{k=1}^{p} \Gamma_{\alpha \beta_k}{}^{\eta} F_{\beta_1 \beta_2 \dots \eta \dots \beta_p} \right) \gamma^{\alpha \beta_1 \beta_2 \dots \beta_p}, \end{aligned} \tag{7.10}$$

where the index η appears in the sequence $\beta_1 \beta_2 \dots \beta_p$ in the kth slot vacated by β_k. It should be apparent that, because of the symmetry of the indices, none of the terms under the summation symbol make any contribution to $\mathbf{d}\mathscr{F}$. For example, by switching the position of α and β_1, we have

$$\Gamma_{\alpha \beta_1}{}^{\eta} \gamma^{\alpha \beta_1 \beta_2 \dots \beta_p} = -\Gamma_{\beta_1 \alpha}{}^{\eta} \gamma^{\beta_1 \alpha \beta_2 \beta_3 \dots \beta_p}.$$

If we then reverse the labels of the dummy indices α and β_1, we have

$$\Gamma_{\beta_1 \alpha}{}^{\eta} \gamma^{\beta_1 \alpha \beta_2 \dots \beta_p} = \Gamma_{\alpha \beta_1}{}^{\eta} \gamma^{\alpha \beta_1 \beta_2 \dots \beta_p}$$

and thus

$$\Gamma_{\alpha \beta_1}{}^{\eta} \gamma^{\alpha \beta_1 \beta_2 \dots \beta_p} = -\Gamma_{\alpha \beta_1}{}^{\eta} \gamma^{\alpha \beta_1 \beta_2 \dots \beta_p} = 0.$$

In a similar fashion,

$$\Gamma_{\alpha\beta_k}{}^{\eta}\gamma^{\alpha\beta_1\beta_2\cdots\beta_k\cdots\beta_p} = 0 \qquad \text{for } k = 1, 2, \ldots, p \tag{7.11}$$

Thus Eq. (7.10) becomes

$$\mathbf{d}\mathscr{F} = \frac{1}{p!}\left(\frac{\partial}{\partial u^\alpha} F_{\beta_1\beta_2\cdots\beta_p}\right)\gamma^{\alpha\beta_1\beta_2\cdots\beta_p} \tag{7.12}$$

where

$$\mathscr{F} = \frac{1}{p!} F_{\beta_1\beta_2\ldots\beta_p}\gamma^{\beta_1\beta_2\cdots\beta_p}.$$

In the formalism of differential forms the analogous equation would be written in the form

$$\mathbf{d}\mathscr{F} = \frac{1}{p!}\left(\frac{\partial}{\partial u^\alpha} F_{\beta_1\beta_2\ldots\beta_p}\right) \mathrm{d}u^\alpha\, \mathrm{d}u^{\beta_1}\, \mathrm{d}u^{\beta_2} \ldots \mathrm{d}u^{\beta_p} \tag{7.13}$$

where

$$\mathscr{F} = \frac{1}{p!} F_{\beta_1\beta_2\ldots\beta_p}\, \mathrm{d}u^{\beta_1}\, \mathrm{d}u^{\beta_2} \ldots \mathrm{d}u^{\beta_p}.$$

Now let us turn to Eq. (7.9). This equation is more difficult to deal with but it too can be cast in a simpler form. We first note that

$$\begin{aligned}
\boldsymbol{\delta}\mathscr{F} &= \frac{1}{(p-1)!}(F^{\alpha\beta_1\beta_2\cdots\beta_{p-1}}{}_{;\alpha}\gamma_{\beta_1\beta_2\ldots\beta_{p-1}}) \\
&= \frac{1}{(p-1)!}\left[\frac{\partial}{\partial u^\alpha} F^{\alpha\beta_1\beta_2\cdots\beta_{p-1}} + \Gamma_{\alpha\eta}{}^{\alpha} F^{\eta\beta_1\beta_2\cdots\beta_{p-1}} \right. \\
&\qquad \left. + \sum_{k=1}^{p} \Gamma_{\alpha\eta}{}^{\beta_k} F^{\alpha\beta_1\beta_2\cdots\eta\ldots\beta_{p-1}}\right]\gamma_{\beta_1\beta_2\cdots\beta_{p-1}}.
\end{aligned} \tag{7.14}$$

It should be observed that $\Gamma_{\alpha\eta}{}^{\beta_k} F^{\alpha\beta_1\beta_2\cdots\beta_{p-1}} = 0$ for essentially the same reason that Eq. (7.11) was shown to be valid.

We are now forced to consider the nature of the contracted Christoffel symbol $\Gamma_{\alpha\eta}{}^{\alpha}$.

$$\Gamma_{\alpha\eta}{}^{\alpha} = \frac{g^{\alpha\nu}}{2}\left(\frac{\partial g_{\nu\eta}}{\partial u^\alpha} + \frac{\partial g_{\nu\alpha}}{\partial u^\eta} - \frac{\partial g_{\alpha\eta}}{\partial u^\nu}\right),$$

and thus

$$\Gamma_{\alpha\eta}{}^{\alpha} = \tfrac{1}{2} g^{\alpha\nu} \frac{\partial g_{\alpha\nu}}{\partial u^{\eta}}. \tag{7.15}$$

This can be simplied further. If $g = \det(g_{\alpha\nu})$, then

$$g = g_{\alpha\nu} \times (\text{the cofactor of } g_{\alpha\nu}) + \text{terms not involving } g_{\alpha\nu}.$$

Therefore

$$\frac{\partial g}{\partial u^{\eta}} = \frac{\partial g_{\alpha\nu}}{\partial u^{\eta}} \times (\text{the cofactor of } g_{\alpha\nu}) + \text{terms not involving } \frac{\partial g_{\alpha\nu}}{\partial u^{\eta}}.$$

To be more precise

$$\frac{\partial g}{\partial u^{\eta}} = \sum_{\alpha,\nu} \left[\frac{\partial g_{\alpha\nu}}{\partial u^{\eta}} \times (\text{the cofactor of } g_{\alpha\nu}) \right].$$

Anyone who has computed the inverse of a matrix should know that the cofactor of $g_{\alpha\nu}$ is $g^{\alpha\nu} g$. Therefore

$$\frac{\partial g}{\partial u^{\eta}} = g^{\alpha\nu} g \frac{\partial g_{\alpha\nu}}{\partial u^{\eta}}$$

and Eq. (7.15) becomes

$$\Gamma_{\alpha\eta}{}^{\alpha} = \frac{1}{2g} \frac{\partial g}{\partial u^{\eta}} = \frac{1}{2|g|} \frac{\partial |g|}{\partial u^{\eta}} = \frac{1}{\sqrt{|g|}} \frac{\partial}{\partial u^{\eta}} \sqrt{|g|}. \tag{7.16}$$

With this result, Eq. (7.14) becomes

$$\boldsymbol{\delta}\mathscr{F} = \frac{1}{(p-1)!} \frac{1}{\sqrt{|g|}} \left(\frac{\partial}{\partial u^{\alpha}} (\sqrt{|g|} F^{\alpha\beta_1\beta_2 \ldots \beta_{p-1}}) \right) \gamma_{\beta_1\beta_2 \ldots \beta_{p-1}} \tag{7.17}$$

where as before

$$\mathscr{F} = \frac{1}{p!} F^{\beta_1\beta_2 \ldots \beta_p} \gamma_{\beta_1\beta_2 \ldots \beta_p}.$$

A simple equation follows almost immediately from Eq. (7.12); that is

$$\mathbf{dd}\mathscr{F} = 0. \tag{7.18}$$

This equation is labeled the *Poincaré lemma* by some authors and the *converse of the Poincaré* **lemma** by others. To verify this equation, we note

that from Eq. (7.12), it follows that

$$\mathbf{d}\mathbf{d}\mathscr{F} = \frac{1}{p!}\frac{\partial^2}{\partial u^\nu\,\partial u^\eta} F_{\beta_1\beta_2\,.\,\beta_p}\gamma^{\nu\eta\beta_1\beta_2\,..\,\beta_p}.$$

Since

$$\frac{\partial^2}{\partial u^\nu\,\partial u^\eta} = \frac{\partial^2}{\partial u^\eta\,\partial u^\nu}$$

and

$$\gamma^{\nu\eta\beta_1\beta_2\ldots\beta_p} = -\gamma^{\eta\nu\beta_1\beta_2\,.\,\beta_p},$$

it immediately follows that $\mathbf{d}\mathbf{d}\mathscr{F} = 0$.

If $\mathscr{F}$ is a p-vector such that $\mathbf{d}\mathscr{F} = 0$, then $\mathscr{F}$ is said to be *closed.* If there exists a $(p-1)$-vector $\mathscr{G}$ such that $\mathscr{F} = \mathbf{d}\mathscr{G}$, $\mathscr{F}$ is said to be *exact.* Obviously if $\mathscr{F}$ is exact then it is also closed (Poincaré's lemma). However, the converse of Poincaré's lemma is true for any region that can be continuously deformed to a point but it may not be true globally (Flanders 1963, pp. 27–29).

Examples of closed 1-vectors are upper index coordinate Dirac matrices. To verify this, we note that

$$\mathbf{d}\gamma^\beta = \gamma^\alpha \wedge \nabla_\alpha\gamma^\beta = -\gamma^\alpha \wedge \Gamma_{\alpha\eta}{}^\beta\gamma^\eta = -\Gamma_{\alpha\eta}{}^\beta\gamma^{\alpha\eta} = 0,$$

since $\Gamma_{\alpha\eta}{}^\beta = \Gamma_{\eta\alpha}{}^\beta$.

Locally any of these upper index coordinate Dirac matrices are exact since

$$\gamma^\alpha = \gamma^\beta\nabla_\beta u^\alpha = \mathbf{d}u^\alpha. \tag{7.19}$$

On the other hand, we may not be able to extend the coordinate u^α over the entire space. Consider for example, the use of polar coordinates for $\mathbb{R}^2$. In that case γ^θ is not defined at $r = 0$. It is true that $\gamma^\theta = \mathbf{d}\theta$, but θ cannot be extended in a continuous manner to cover $\mathbb{R}^2$ even if the point where $r = 0$ is omitted. (See Fig. 7.1.)

A theorem analogous to the Poincaré lemma also applies to the codifferential operator $\boldsymbol{\delta}$; that is

$$\boldsymbol{\delta\delta}\mathscr{F} = 0. \tag{7.20}$$

To verify Eq. (7.20); we will first demonstrate a relation between $\boldsymbol{\delta}$ and $\mathbf{d}$ that can be expressed simply in terms of the *normalized pseudo-scalar J.*

The n-vector J can be expressed most simply in terms of orthonormal Dirac matrices. In particular

$$J = \hat{\gamma}_1\hat{\gamma}_2\cdots\hat{\gamma}_n = \hat{\gamma}_{12\,\ldots\,n}. \tag{7.21}$$

Since the space of n-vectors is 1-dimensional, Eq. (7.21) defines J uniquely.

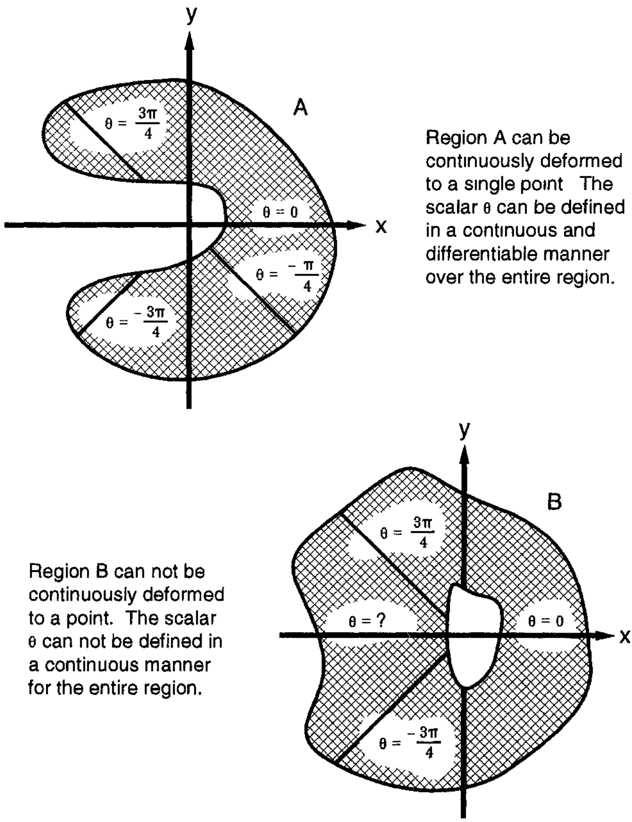


Fig. 7.1.

It should be clear that $\hat{\gamma}_j$ commutes with J if n is odd and anticommutes with J if n is even. This is true because $\hat{\gamma}_j$ commutes with itself and anticommutes with each of the remaining $(n-1)$ $\hat{\gamma}_i$'s that are used to form the product J. Since any 1-vector γ^k is a linear combination of the orthonormal $\hat{\gamma}_j$'s, we have

$$\gamma^k J = (-1)^{n-1} J \gamma^k. \tag{7.22}$$

Furthermore

$$\nabla_k J = 0. \tag{7.23}$$

To show this is true, we note that

$$\nabla_k J = \partial_k J - \mathbf{\Gamma}_k J + J\mathbf{\Gamma}_k = -\mathbf{\Gamma}_k J + J\mathbf{\Gamma}_k.$$

From Eq. (7.22), it is clear that J commutes with any p-vector of even order. Since $\mathbf{\Gamma}_k$ is a 2-vector, J must commute with $\mathbf{\Gamma}_k$ and thus $\nabla_k J = 0$.

Taken together, Eqs. (7.22) and (7.23) imply that

$$J\nabla J\mathscr{F} = J(\gamma^k\nabla_k)J\mathscr{F} = (-1)^{n-1}(J)^2\nabla\mathscr{F}. \tag{7.24}$$

Now it is not difficult to show that

$$(J)^2 = (-1)^s(-1)^{n(n-1)/2}I, \tag{7.25}$$

where s is the number of -1's on the diagonal of the signature matrix. (See Problem 7.3.) When this result is applied to Eq. (7.24), we get

$$J\nabla J\mathscr{F} = (-1)^{s+1}(-1)^{n(n+1)/2}\nabla\mathscr{F}$$

and therefore,

$$\mathbf{d}\mathscr{F} + \boldsymbol{\delta}\mathscr{F} = (-1)^{s+1}(-1)^{n(n+1)/2}(J\mathbf{d}J\mathscr{F} + J\boldsymbol{\delta}J\mathscr{F}). \tag{7.26}$$

Now if $\mathscr{F}$ is a p-vector, then $J\mathscr{F}$ is an $(n-p)$-vector, $\mathbf{d}J\mathscr{F}$ is an $(n-p+1)$-vector, and finally $J\mathbf{d}J\mathscr{F}$ is an $n-(n-p+1)$ or $(p-1)$-vector. A similar walk through the computation of $J\boldsymbol{\delta}J\mathscr{F}$ reveals that it is a $(p+1)$-vector. If we now equate the $(p+1)$-vectors and the $(p-1)$-vectors that appear on the two sides of Eq. (7.26), we have

$$\mathbf{d}\mathscr{F} = (-1)^{s+1}(-1)^{n(n+1)/2}J\boldsymbol{\delta}J\mathscr{F}, \tag{7.27}$$

and

$$\boldsymbol{\delta}\mathscr{F} = (-1)^{s+1}(-1)^{n(n+1)/2}J\mathbf{d}J\mathscr{F}. \tag{7.28}$$

From Eqs. (7.28) and (7.25), we now have

$$\boldsymbol{\delta\delta}\mathscr{F} = (J\mathbf{d}J)(J\mathbf{d}J)\mathscr{F} = J\mathbf{d}(J)^2\mathbf{d}J\mathscr{F} = (-1)^s(-1)^{n(n-1)/2}J\mathbf{dd}(J\mathscr{F}) = 0.$$

In this fashion, we have verified Eq. (7.20).

In closing this section, I would like to draw your attention to two items if you are already familiar with the formalism of differential forms. One: in section 3.3 of Chapter 3, we noted that the upper index Dirac matrices are the Clifford algebra analogues of the 1-forms that appear in the formalism of differential forms. Now that we have introduced the exterior derivative $\mathbf{d}$, this point arises again. This is particularly true when we compare Eq. (7.12) with Eq. (7.13). It also crops up in Eq. (7.19). Second: multiplication by the normalized pseudoscalar J is essentially identical to applying the Hodge star operator (Flanders 1963, pp. 15–17). Except for some differences in sign conventions, the two entities considered as operators are identical. However, for computational purposes it is much easier to use the operator J in the context of Clifford algebra than it is to use the Hodge star operator in the

context of differential forms. We will take advantage of this to compute the Kerr metric in Chapter 9.

Problem 7.1. Multiply both sides of Eq. (7.3) by $W^{\alpha}_{k} W_{j_1}{}^{\beta_1} W_{j_2}{}^{\beta_2} \dots W_{j_p}{}^{\beta_p}$ to show that

$$\gamma^{\alpha}\gamma^{\beta_1\beta_2\dots\beta_p} = \gamma^{\alpha\beta_1\beta_2\dots\beta_p} + \sum_{k=1}^{p} (-1)^{k-1} g^{\alpha\beta_k}\gamma^{\beta_1\beta_2\dots\hat{\beta}_k\dots\beta_p}.$$

(This should convince you that Eq. (7.4) is also valid for noncoordinate bases. The proof of Eq. (7.4) in its full generality is awkward only because of the limitations of the notation chosen for this text. Mathematicians need a few more alphabets to do their thing.)

Problem 7.2. Show that if $\mathscr{F}$ is a p-vector, then $\mathscr{F}^{\dagger} = (-1)^{p(p-1)/2}\mathscr{F}$.

Problem 7.3. Show that $(J)^2 = (-1)^s(-1)^{n(n-1)/2}I$, where s is the number of -1's on the diagonal of the signature matrix. (Suggestion: compute $JJ^{\dagger}$ and use the result of Problem 7.2.)

Problem 7.4. Suppose that $\mathscr{F}$ is a twice differentiable Clifford number.

(1) Show that $\boldsymbol{\nabla}^2\mathscr{F} = (\mathbf{d}\boldsymbol{\delta} + \boldsymbol{\delta}\mathbf{d})\mathscr{F}$.
(2) Show that if $\mathscr{F}$ is a p-vector then $\boldsymbol{\nabla}^2\mathscr{F}$ is also a p-vector.

Problem 7.5. Show that if $\mathscr{F}$ is a differentiable p-vector and $\mathscr{G}$ is any differentiable Clifford number, then

$$\mathbf{d}\mathscr{F} \wedge \mathscr{G} = (\mathbf{d}\mathscr{F}) \wedge \mathscr{G} + (-1)^p\mathscr{F} \wedge \mathbf{d}\mathscr{G}. \tag{7.29}$$

7.2 Maxwell's Equations in Flat Space

In the units of Heaviside–Lorentz, Maxwell's equations are as follows:

$$\vec{\nabla}\cdot\vec{B} = 0, \tag{7.30}$$

$$\frac{1}{c}\frac{\partial\vec{B}}{\partial t} + \vec{\nabla}\times\vec{E} = 0, \tag{7.31}$$

$$\vec{\nabla}\cdot\vec{E} = \rho, \tag{7.32}$$

$$-\frac{1}{c}\frac{\partial\vec{E}}{\partial t} + \vec{\nabla}\times\vec{B} = \frac{1}{c}\vec{J}. \tag{7.33}$$

Furthermore, if $\partial/\partial t$ is applied to both sides of Eq. (7.32) and $\vec{\nabla}\cdot$ is applied

to both sides of Eq. (7.33) and the two resulting equations are added, one obtains

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot\vec{J} = 0. \tag{7.34}$$

A related equation is the equation of motion for a small test particle due to the Lorentz force. That is

$$mc^2 \frac{d\vec{u}}{ds} = (q(\vec{E} + \vec{u} \times \vec{B}), \tag{7.35}$$

where

$$\vec{u} = (u^1, u^2, u^3) = \left(\frac{v_x/c}{\sqrt{1-(v/c)^2}}, \frac{v_y/c}{\sqrt{1-(v/c)^2}}, \frac{v_z/c}{\sqrt{1-(v/c)^2}}\right),$$

and q is the charge of the test particle.

To formulate these equations in terms of Clifford algebra, we first introduce a coordinate system of Dirac matrices. Let

$$x^0 = ct, \qquad x^1 = x, \qquad x^2 = y, \qquad x^3 = z;$$

$$(\gamma_0)^2 = -(\gamma_1)^2 = -(\gamma_2)^2 = -(\gamma_3)^2 = I;$$

and

$$\gamma_\alpha\gamma_\beta + \gamma_\beta\gamma_\alpha = 0 \quad \text{for } \alpha \neq \beta.$$

It is also useful to introduce the *Faraday 2-vector* $\mathscr{F} = \frac{1}{2}F_{\alpha\beta}\gamma^{\alpha\beta}$ which is defined by the relation

$$[F_{\alpha\beta}] = \begin{bmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{bmatrix}. \tag{7.36}$$

With this definition, it is not difficult to show that Eqs. (7.30) and (7.31) can be combined into the single equation

$$\mathbf{d}\mathscr{F} = 0. \tag{7.37}$$

The signs for the Faraday 2-vector depend on the signature of the metric tensor. Whichever signature is used, the sign is generally chosen so that the equation for the Lorentz force can be incorporated into an equation written in the form

$$mc^2 a^\alpha = qF^\alpha{}_\beta u^\beta, \tag{7.38}$$

where m is the mass of the particle and q is the charge of the test particle.

The tensor character of $F_{\alpha\beta}$ serves to underline the intimate connection between the magnetic and electric fields. An observer who is stationary with respect to a system of charges which are stationary with respect to one another will observe an electric field only. However, an observer in another coordinate system which is moving with respect to the first will observe both a magnetic and an electric field.

To compute the codifferential of $\mathscr{F}$, we note that from Eq. (7.17) for flat space,

$$\boldsymbol{\delta}\mathscr{F} = \left(\frac{\partial}{\partial x^\alpha} F^{\alpha\beta}\right)\gamma_\beta. \tag{7.39}$$

To examine the detailed structure of Eq. (7.39), we need the contravariant components of the Faraday tensor. These can be computed from Eq. (7.36). The result is

$$[F^{\alpha\beta}] = \begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{bmatrix}, \tag{7.40}$$

From Eqs. (7.39) and (7.40) one gets

$$\begin{aligned} \boldsymbol{\delta}\mathscr{F} &= \left(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}\right)\gamma_0 + \left(\frac{-\partial E_x}{c\,\partial t} + \frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z}\right)\gamma_1 \\ &\quad + \left(\frac{-\partial E_y}{c\,\partial t} - \frac{\partial B_z}{\partial x} + \frac{\partial B_x}{\partial z}\right)\gamma_2 + \left(\frac{-\partial E_z}{c\,\partial t} + \frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y}\right)\gamma_3. \end{aligned}$$

Using Eqs. (7.32) and (7.33), this becomes

$$\boldsymbol{\delta}\mathscr{F} = \left(\rho\gamma_0 + \frac{J_x}{c}\gamma_1 + \frac{J_y}{c}\gamma_2 + \frac{J_z}{c}\gamma_3\right).$$

Thus it is natural to introduce the *current 1-vector*

$$\mathscr{J} = J^\nu\gamma_\nu = \rho\gamma_0 + \frac{J_x}{c}\gamma_1 + \frac{J_y}{c}\gamma_2 + \frac{J_z}{c}\gamma_3. \tag{7.41}$$

With the introduction of this current 1-form, Eqs. (7.32) and (7.33) can be

combined into the single equation

$$\boldsymbol{\delta}\mathscr{F} = \mathscr{I}. \tag{7.42}$$

Writing out Maxwell's equations in terms of vector components requires eight equations. Using vector notation, it is possible to reduce the number of equations to four. We see that using the formalism of differential forms, only two equations are necessary (Eqs. (7.37) and (7.42)). However, using the language of Clifford algebra, it is possible to condense the entire system into a single equation. Since $\boldsymbol{\nabla}\mathscr{F} = \mathbf{d}\mathscr{F} + \boldsymbol{\delta}\mathscr{F}$, we have

$$\boldsymbol{\nabla}\mathscr{F} = \mathscr{J}. \tag{7.43}$$

Since $\mathbf{d}\mathscr{F} = 0$, we know from the converse of Poincaré's lemma that there exists a *vector potential* $\mathscr{A}$ such that at least locally

$$\mathscr{F} = \mathbf{d}\mathscr{A}. \tag{7.44}$$

Writing out the components of Eq. (7.44), we have

$$\begin{aligned}
\mathscr{F} = \mathbf{d}\mathscr{A} &= \left(\frac{\partial A_0}{\partial x^1} - \frac{\partial A_1}{\partial x^0}\right)\gamma^{10} + \left(\frac{\partial A_0}{\partial x^2} - \frac{\partial A_2}{\partial x^0}\right)\gamma^{20} \\
&\quad + \left(\frac{\partial A_0}{\partial x^3} - \frac{\partial A_3}{\partial x^0}\right)\gamma^{30} + \left(\frac{\partial A_3}{\partial x^2} - \frac{\partial A_2}{\partial x^3}\right)\gamma^{23} \\
&\quad + \left(\frac{\partial A_1}{\partial x^3} - \frac{\partial A_3}{\partial x^1}\right)\gamma^{31} + \left(\frac{\partial A_2}{\partial x^1} - \frac{\partial A_1}{\partial x^2}\right)\gamma^{12} \\
&= -E_x\gamma^{10} - E_y\gamma^{20} - E_z\gamma^{30} \\
&\quad - B_x\gamma^{23} - B_y\gamma^{31} - B_z\gamma^{12}.
\end{aligned}$$

Since

$$\vec{E} = -\vec{\nabla}\phi - \frac{\partial\vec{A}}{c\,\partial t} \qquad \text{and} \qquad \vec{B} = \vec{\nabla} \times \vec{A},$$

we see that

$$\mathscr{A} = A_\alpha\gamma^\alpha = \phi\gamma^0 - A_x\gamma^1 - A_y\gamma^2 - A_z\gamma^3, \tag{7.45}$$

or

$$\mathscr{A} = A^\alpha\gamma_\alpha = \phi\gamma_0 + A_x\gamma_1 + A_y\gamma_2 + A_z\gamma_3. \tag{7.46}$$

(Regardless of the signature used, the signs in the equations are adjusted so that $(A^1, A^2, A^3) = (A_x, A_y, A_z)$.)

Summarizing our equations, we have

$$\mathscr{F} = \mathbf{d}\mathscr{A}, \tag{7.47}$$

$$\mathbf{d}\mathscr{F} = \mathbf{dd}\mathscr{A} = 0, \tag{7.48}$$

$$\boldsymbol{\delta}\mathscr{F} = \boldsymbol{\delta}\mathbf{d}\mathscr{A} = \mathscr{J}. \tag{7.49}$$

It is sometimes useful to divide electromagnetic phenomena into two categories—those which are caused by free charges and free currents and those which are caused by bound charges and bound currents:

$$\mathscr{J} = \mathscr{J}_{\text{FREE}} + \mathscr{J}_{\text{BOUND}} \tag{7.50}$$

and

$$\mathscr{F} = \mathscr{F}' + \mathscr{M}. \tag{7.51}$$

It is understood here that

$$\mathscr{J}_{\text{FREE}} = \boldsymbol{\delta}\mathscr{F}' \tag{7.52}$$

and

$$\mathscr{J}_{\text{BOUND}} = \boldsymbol{\delta}\mathscr{M}. \tag{7.53}$$

If $\mathscr{F}' = \frac{1}{2}F'^{\alpha\beta}\gamma_{\alpha\beta}$, then

$$[F'^{\alpha\beta}] = \begin{bmatrix} 0 & -D_x & -D_y & -D_z \\ D_x & 0 & -H_z & H_y \\ D_y & H_z & 0 & -H_x \\ D_z & -H_y & H_x & 0 \end{bmatrix}. \tag{7.54}$$

In addition, if $\mathscr{M} = \frac{1}{2}M^{\alpha\beta}\gamma_{\alpha\beta}$, then

$$[M^{\alpha\beta}] = \begin{bmatrix} 0 & P_x & P_y & P_z \\ -P_x & 0 & -M_z & M_y \\ -P_y & M_z & 0 & -M_x \\ -P_z & -M_y & M_x & 0 \end{bmatrix}. \tag{7.55}$$

In standard books on electricity and magnetism, Eq. (7.51) appears as

two equations, namely

$$\vec{E} = \vec{D} - \vec{P} \tag{7.56}$$

and

$$\vec{B} = \vec{H} + \vec{M}. \tag{7.57}$$

Problem 7.6. Since $\mathscr{J} = \boldsymbol{\delta}\mathscr{F}$, it immediately follows that $\boldsymbol{\delta}\mathscr{J} = 0$. Compare this result with Eq. (7.34).

Problem 7.7. The equation $\boldsymbol{\delta}\mathbf{d}\mathscr{A} = \mathscr{J}$ is usually solved for $\mathscr{A}$ with the subsidiary Lorentz condition that

$$\vec{\nabla}\cdot\vec{A} + \frac{\partial\phi}{c\,\partial t} = 0.$$

Show that this is equivalent to the condition that $\boldsymbol{\delta}\mathscr{A} = 0$. Also show that with the Lorentz condition, Eq. (7.49), can be rewritten in the form $\nabla^2\mathscr{A} = \mathscr{J}$.

Problem 7.8.

(1) Show that the Lorentz force Eq. (7.38) can be rewritten in the form

$$\dot{\boldsymbol{u}}(s) = \nabla_s \boldsymbol{u}(s) = \frac{q}{2mc^2}(\mathscr{F}(s)\boldsymbol{u}(s) - \boldsymbol{u}(s)\mathscr{F}(s)), \tag{7.58}$$

where $\mathscr{F}$ is the Faraday 2-vector.

(2) Since the world velocity vector does not change length, it undergoes a generalized rotation or Lorentz transformation. That is

$$\boldsymbol{u}(s) = \mathscr{L}(s)\boldsymbol{u}(0)\mathscr{L}^{-1}(s). \tag{7.59}$$

Use Eqs. (7.58) and (7.59) to show that

$$\nabla_s\mathscr{L}(s) = \frac{q}{2mc^2}\mathscr{F}(s)\mathscr{L}(s). \tag{7.60}$$

(If you get stuck, you may wish to review the derivation of Eq. (5.78).)

(3) Suppose $\mathscr{F}(s) = \mathscr{F}(0)$. Show

$$\mathscr{L}(s) = \exp\left(\frac{sq}{2mc^2}\mathscr{F}\right). \tag{7.61}$$

(4) Suppose $\mathscr{F}(s) = \mathscr{F}(0)$ and the electric field $\vec{E} = 0$. Show

$$\mathscr{L}(s) = I\cos\left(\frac{sqB}{2mc^2}\right) - \sin\left(\frac{sqB}{2mc^2}\right)\left(\frac{B_x}{B}\gamma^{23} + \frac{B_y}{B}\gamma^{31} + \frac{B_z}{B}\gamma^{12}\right), \quad (7.62)$$

where $B = (\vec{B}\cdot\vec{B})^{\frac{1}{2}}$.

Problem 7.9.

(1) Show that

$$\tfrac{1}{2}(\mathscr{F}\mathscr{J} - \mathscr{J}\mathscr{F}) = \gamma_0\left(\frac{\vec{J}\cdot\vec{E}}{c}\right) + \sum_{k=1}^{3}\gamma_k\left(\rho E_k + \frac{(\vec{J}\times\vec{B})_k}{c}\right).$$

(2) Show that

$$(\mathscr{F}\gamma^j\mathscr{F})_{;j} = \mathscr{F}\mathscr{J} - \mathscr{J}\mathscr{F}.$$

(3) Suppose $T^{jk} = \frac{1}{2}(\gamma^j\mathscr{F}\gamma^k\mathscr{F})_0$. Show that $T^{jk} = T^{kj}$. (If you wish to cheat, look at Problem 3.18.)

(3) Show

$$T^{jk}{}_{;k} = \tfrac{1}{2}(\gamma^j(\mathscr{F}\mathscr{J} - \mathscr{J}\mathscr{F}))_0 = \langle\gamma^j, \tfrac{1}{2}(\mathscr{F}\mathscr{J} - \mathscr{J}\mathscr{F})\rangle.$$

(This last result combined with the result of part (1) in effect shows that

$$T^{0k}{}_{;k} = \frac{\vec{J}\cdot\vec{E}}{c} \qquad \text{and} \qquad T^{ik}{}_{;k} = \rho E^i + \frac{(\vec{J}\times\vec{B})^i}{c}$$

for $i = 1, 2,$ and 3. The tensor T^{jk} is known as the *electromagnetic-momentum tensor.*)

Problem 7.10.

(1) Use Eq. (7.49) to show that

$$A^{k;j}{}_{;j} - A^{j;k}{}_{;j} = J^k.$$

(2) Use the result of Problem 5.18 and the Lorentz condition to show that in flat space $A^{k;j}{}_{;j} = J^k$.

Problem 7.11.

(1) Consider the Lagrangian

$$L = \tfrac{1}{2}mc^2 g_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta + q\dot{x}^\alpha A_\alpha,$$

where $\dot{x} = \mathrm{d}x^\alpha/\mathrm{d}s$. The corresponding Euler–Lagrange equations are

$$\frac{\mathrm{d}}{\mathrm{d}s}\frac{\mathrm{d}L}{\partial x^\lambda} = 0 \quad \text{for } \lambda = 0, 1, 2, \text{ or } 3.$$

Show that these equations are the component equations for the Lorentz force; that is

$$mc^2(\ddot{x}^\nu + \Gamma_{\alpha\beta}{}^\nu \dot{x}^\alpha \dot{x}^\beta) = qF^\nu{}_\alpha \dot{x}^\alpha. \tag{7.63}$$

(Comparing Eq. (7.63) with (7.38), one might notice an apparent discrepancy. If $a^\nu = \ddot{x}^\nu$, then the equations would be at odds. However,

$$\boldsymbol{a} = a^\gamma \gamma_\gamma = \nabla_s(\dot{x}^\alpha \gamma_\alpha) = \ddot{x}^\alpha \gamma_\alpha + \dot{x}^\alpha \dot{x}^\beta \nabla_\beta \gamma_\alpha = \ddot{x}^\nu \gamma_\nu + \dot{x}^\alpha \dot{x}^\beta \Gamma_{\beta\alpha}{}^\nu \gamma_\nu.$$

Thus $a^\nu = \ddot{x}^\nu + \Gamma_{\beta\alpha}{}^\nu \dot{x}^\alpha \dot{x}^\beta$.

(2) Show that the corresponding Hamiltonian is

$$H = \dot{x}^\alpha \frac{\partial L}{\partial \dot{x}^\alpha} - L = \tfrac{1}{2}mc^2 u^\alpha u_\alpha = \tfrac{1}{2}mc^2. \tag{7.64}$$

7.3 Is Gravity a Yang–Mills Field?

Those attempting to construct a unified field theory look for similarities in the equations which characterize the four fundamental forces of nature; the electromagnetic, the weak, the strong and the gravitational. The gravitational force seems to be unique in that unlike the other forces it is not considered to be a Yang–Mills field. Nonetheless, in the formalism of Clifford algebra, there are very strong similarities. These similarities are so strong that one is tempted to claim that gravity is a Yang–Mills field.

What is a Yang–Mills field? For the strong and weak forces, the components of the vector potential used in electromagnetic theory are replaced by matrices referred to as *connections* or components of a *gauge potential*. In addition, the Faraday tensor $F_{\alpha\beta}$ is replaced by a *field strength tensor*. In particular

$$F_{\alpha\beta} = \frac{\partial A_\beta}{\partial x^\alpha} - \frac{\partial A_\alpha}{\partial x^\beta} + \mathrm{i}q[A_\alpha, A_\beta], \tag{7.65}$$

where

$$[A_\alpha, A_\beta] = A_\alpha A_\beta - A_\beta A_\alpha.$$

Of course for the electromagnetic case, $[A_\alpha, A_\beta] = 0$.

In this context, one introduces a covariant derivative ∇_α. Acting on a

vector field ϕ, we have

$$\nabla_\alpha \phi = \frac{\partial \phi}{\partial x^\alpha} + \mathrm{i} q A_\alpha \phi.$$

On the other hand, when ∇_α acts on some matrix field M, we have

$$\nabla_\alpha M = \frac{\partial}{\partial x^\alpha} M + \mathrm{i} q [A_\alpha, M]. \tag{7.66}$$

Using Eqs. (7.65) and (7.66), one can prove a Bianchi identity:

$$\nabla_\alpha F_{\mu\nu} + \nabla_\mu F_{\nu\alpha} + \nabla_\nu F_{\alpha\mu} = 0. \tag{7.67}$$

Note: Sign conventions vary, but for the case of the electromagnetic field applied to Dirac's equation for the electron, one writes

$$\nabla_\alpha \psi = \frac{\partial}{\partial x^\alpha} \psi + \frac{\mathrm{i}e}{hc} A_\alpha \psi.$$

We have chosen a sign convention that appears most compatible with this special case of a Yang–Mills field.

In contrast to Eq. (7.66), some textbook authors define

$$\nabla_\alpha = \frac{\partial}{\partial x^\alpha} + \mathrm{i} q A_\alpha. \tag{7.68}$$

Without additional multiplication conventions, this definition is incompatible with the Bianchi identity. (Also see Problem 7.12.)

For the electromagnetic case, Eq. (7.67) reduces to

$$\frac{\partial}{\partial x^\alpha} F_{\mu\nu} + \frac{\partial}{\partial x^\mu} F_{\nu\alpha} + \frac{\partial}{\partial x^\nu} F_{\alpha\mu} = 0. \tag{7.69}$$

Equation (7.69) is equivalent to Eq. (7.37) which is also known as the homogeneous set of Maxwell's equations.

Generally the equation for the gauge field A_α or the field strength tensor in terms of a source term J^α is derived by carrying out a variation on the action

$$\begin{aligned} S &= \int \mathrm{Tr}(-\tfrac{1}{4} F_{\alpha\beta} F^{\alpha\beta} + J^\alpha A_\alpha)\sqrt{-g}\, \mathrm{d}x^0\, \mathrm{d}x^1\, \mathrm{d}x^2\, \mathrm{d}x^3 \\ &= \int \mathrm{Tr}(-\tfrac{1}{4} F_{\alpha\beta} g^{\alpha\eta} g^{\beta\nu} F_{\eta\nu} + J^\alpha A_\alpha)\sqrt{-g}\, \mathrm{d}x^0\, \mathrm{d}x^1\, \mathrm{d}x^2\, \mathrm{d}x^3 \end{aligned} \tag{7.70}$$

where Tr(M) is the trace of the matrix M. We should note that J^α is generally constructed from interacting fields.

To carry out the variation, we see that

$$\delta S = \int \mathrm{Tr}(-\tfrac{1}{4}(\delta F_{\alpha\beta})g^{\alpha\eta}g^{\beta\nu}F_{\eta\nu} + F_{\alpha\beta}g^{\alpha\eta}g^{\beta\nu}\delta F_{\eta\nu} + J^\alpha \delta A_\alpha)\sqrt{-g}\,\mathrm{d}x^0\,\mathrm{d}x^1\,\mathrm{d}x^2\,\mathrm{d}x^3 \tag{7.71}$$

where

$$F_{\alpha\beta} = \frac{\partial A_\beta}{\partial x^\alpha} - \frac{\partial A_\alpha}{\partial x^\beta} + \mathrm{i}q[A_\alpha, A_\beta]$$

and

$$\begin{aligned}\delta F_{\alpha\beta} = {} & \delta^\theta_\beta \frac{\partial}{\partial x^\alpha}\delta A_\theta - \delta^\theta_\alpha \frac{\partial}{\partial x^\beta} A_\theta \\ & + \mathrm{i}q\delta^\theta_\alpha \delta A_\theta A_\beta + \mathrm{i}q\,\delta^\theta_\beta A_\alpha \delta A_\theta \\ & - \mathrm{i}q\delta^\theta_\beta \delta A_\theta A_\alpha - \mathrm{i}q\delta^\theta_\alpha A_\beta \delta A_\theta.\end{aligned}$$

When these terms are plugged back into Eq. (7.71) they must be sorted out with some care since the matrices do not commute. The first term that occurs in Eq. (7.71) is

$$\begin{aligned}&\int \mathrm{Tr}\left(-\tfrac{1}{4}\delta^\theta_\beta\left(\frac{\partial}{\partial x^\alpha}\delta A_\theta\right)g^{\alpha\eta}g^{\beta\nu}F_{\eta\nu}\sqrt{-g}\right)\mathrm{d}x^0\,\mathrm{d}x^1\,\mathrm{d}x^2\,\mathrm{d}x^3 \\ &\qquad = \int \mathrm{Tr}\left(-\frac{1}{4}\left(\frac{\partial}{\partial x^\alpha}\delta A_\theta\right)F^{\alpha\theta}\sqrt{-g}\right)\mathrm{d}x^0\,\mathrm{d}x^1\,\mathrm{d}x^2\,\mathrm{d}x^3.\end{aligned}$$

Integrating by parts, this becomes

$$-\frac{1}{4}\int \frac{\partial}{\partial x^\alpha}\mathrm{Tr}(\delta A_\theta F^{\alpha\theta}\sqrt{-g})\,\mathrm{d}x^0\,\mathrm{d}x^1\,\mathrm{d}x^2\,\mathrm{d}x^3 + \frac{1}{4}\int \mathrm{Tr}\left(\delta A_\theta \frac{\partial}{\partial x^\alpha}(F^{\alpha\theta}\sqrt{-g})\right)\mathrm{d}x^0\,\mathrm{d}x^1\,\mathrm{d}x^2\,\mathrm{d}x^3. \tag{7.72}$$

The first integral is zero if we assume $\delta A_\theta = 0$ on the boundary of our region of integration. Since $F^{\alpha\theta} = -F^{\theta\alpha}$, we can rewrite the second integral in the form

$$-\frac{1}{4}\int \mathrm{Tr}\left(\delta A_\theta \frac{\partial}{\partial x^\alpha}(F^{\theta\alpha}\sqrt{-g})\right)\mathrm{d}x^0\,\mathrm{d}x^1\,\mathrm{d}x^2\,\mathrm{d}x^3.$$

Another term that occurs in Eq. (7.71) is

$$-\frac{1}{4}\int \mathrm{Tr}(\mathrm{i}q\delta^{\theta}_{\beta}A_{\alpha}\delta A_{\theta}g^{\alpha\eta}g^{\beta\nu}F_{\eta\nu})\sqrt{-g}\,\mathrm{d}x^0\,\mathrm{d}x^1\,\mathrm{d}x^2\,\mathrm{d}x^3$$
$$= -\frac{1}{4}\int \mathrm{Tr}(\mathrm{i}qA_{\alpha}\delta A_{\theta}F^{\alpha\theta}\sqrt{-g})\,\mathrm{d}x^0\,\mathrm{d}x^1\,\mathrm{d}x^2\,\mathrm{d}x^3.$$

Since $\mathrm{Tr}(ABC) = \mathrm{Tr}(CAB) = \mathrm{Tr}(BCA)$, this last term can be rewritten in the form

$$-\frac{1}{4}\int \mathrm{Tr}(\delta A_{\theta}(\mathrm{i}qF^{\alpha\theta}\sqrt{-g}A_{\alpha})\,\mathrm{d}x^0\,\mathrm{d}x^1\,\mathrm{d}x^2\,\mathrm{d}x^3$$
$$= +\frac{1}{4}\int \mathrm{Tr}(\delta A_{\theta}(\mathrm{i}qF^{\theta\alpha}\sqrt{-g}A_{\alpha})\,\mathrm{d}x^0\,\mathrm{d}x^1\,\mathrm{d}x^2\,\mathrm{d}x^3.$$

When all the terms in Eq. (7.71) are dealt with in a similar manner and like terms are collected, we get

$$\delta S = -\int \mathrm{Tr}\left(\delta A_{\theta}\left(\frac{\partial}{\partial x^{\alpha}}(F^{\theta\alpha}\sqrt{-g}) + \mathrm{i}qA_{\alpha}(F^{\theta\alpha}\sqrt{-g})\right.\right.$$
$$= \left.\left. - \mathrm{i}q(F^{\theta\alpha}\sqrt{-g})A_{\alpha} - J^{\theta}\sqrt{-g}\right)\right)\mathrm{d}x^0\,\mathrm{d}x^1\,\mathrm{d}x^2\,\mathrm{d}x^3.$$

This implies that

$$\frac{\partial}{\partial x^{\alpha}}(F^{\theta\alpha}\sqrt{-g}) + \mathrm{i}qA_{\alpha}(F^{\theta\alpha}\sqrt{-g}) - \mathrm{i}q(F^{\theta\alpha}\sqrt{-g})A_{\alpha} = J^{\theta}\sqrt{-g}$$

or

$$\frac{1}{\sqrt{-g}}\nabla_{\alpha}(F^{\theta\alpha}\sqrt{-g}) = J^{\theta}. \tag{7.73}$$

In the electromagnetic case, this becomes

$$\frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^{\alpha}}(F^{\theta\alpha}\sqrt{-g}) = J^{\theta}.$$

According to Eqs. (7.9) and (7.17), this is equivalent to the equation

$$F^{\theta\alpha}{}_{;\alpha} = J^{\theta}. \tag{7.74}$$

For the electromagnetic case, J^θ must satisfy a continuity equation. That is

$$J^\theta{}_{,\theta} = \frac{\partial}{\partial x^\theta} J^\theta + J^\beta \Gamma_{\theta\beta}{}^\theta = \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\theta} (\sqrt{-g} J^\theta)$$
$$= \frac{1}{\sqrt{-g}} \frac{\partial^2}{\partial x^\theta \, \partial x^\alpha} (F^{\theta\alpha} \sqrt{-g}).$$

This last term equals zero because of the symmetry of the dummy indices. Therefore for the electromagnetic case

$$\frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\theta} (\sqrt{-g} J^\theta) = 0. \tag{7.75}$$

For the more general Yang–Mills equation, we can show that

$$\frac{1}{\sqrt{-g}} \nabla_\theta (\sqrt{-g} J^\theta) = 0. \tag{7.76}$$

To prove Eq. (7.76), we first note that it can be shown that

$$(\nabla_\alpha \nabla_\beta - \nabla_\beta \nabla_\alpha) M = \mathrm{i}q \left(\frac{\partial A_\beta}{\partial x^\alpha} - \frac{\partial A_\alpha}{\partial x^\beta} + \mathrm{i}q[A_\alpha, A_\beta] \right) M$$
$$- \mathrm{i}qM \left(\frac{\partial A_\beta}{\partial x^\alpha} - \frac{\partial A_\beta}{\partial x^\alpha} + \mathrm{i}q[A_\alpha, A_\beta] \right)$$
$$= \mathrm{i}q[F_{\alpha\beta}, M]. \tag{7.77}$$

To prove Eq. (7.77), we merely apply Eq. (7.66) twice and then discover that a lot of terms cancel out. In turn, Eq. (7.77) implies that

$$(\nabla_\alpha \nabla_\beta - \nabla_\beta \nabla_\alpha)(f F^{\alpha\beta}) = \mathrm{i}q[F_{\alpha\beta}, f F^{\alpha\beta}]$$
$$= \mathrm{i}qf(F_{\alpha\beta} F^{\alpha\beta} - F^{\alpha\beta} F_{\alpha\beta})$$
$$= \mathrm{i}qf(F^{\eta\nu} g_{\eta\alpha} g_{\nu\beta} F^{\alpha\beta} - F^{\alpha\beta} F_{\alpha\beta})$$
$$= \mathrm{i}qf(F^{\eta\nu} F_{\eta\nu} - F^{\alpha\beta} F_{\alpha\beta})$$
$$= 0.$$

Because of the symmetry of the dummy indices, this implies that

$$\nabla_\alpha \nabla_\beta (f F^{\alpha\beta}) = 0. \tag{7.78}$$

A special case of this result occurs when $f = \sqrt{-g}$. Thus

$$\nabla_\theta \nabla_\alpha (\sqrt{-g} F^{\theta\alpha}) = 0$$

or

$$\nabla_\theta \left\{ \sqrt{-g} \left[\frac{1}{\sqrt{-g}} \nabla_\alpha (\sqrt{-g} F^{\theta\alpha}) \right] \right\} = \nabla_\theta (\sqrt{-g} J^\theta) = 0.$$

In the usual presentation, Einstein's field equations have a quite different appearance. Elsewhere in this text, we only deal with the vacuum equations:

$$R_{\alpha\beta} = 0. \tag{7.79}$$

For the more general situation, we have

$$R_{\alpha\beta} = \tfrac{1}{2} R g_{\alpha\beta} = 8\pi T_{\alpha\beta} \tag{7.80}$$

where $T_{\alpha\beta}$ is known as the *energy–momentum tensor.*

Because of the nature of the left-hand side of Eq. (7.80), it is required that $T^{\alpha\beta}{}_{,\beta} = 0$. This is usually regarded as the analogue of the electromagnetic equation $J^\theta{}_{;\theta} = 0$.

It is important to note that when $T_{\alpha\beta} = 0$, $R^\beta{}_\beta - \tfrac{1}{2} R \delta^\beta_\beta = R - 2R = 0$ so $R = 0$ and Eq. (7.80) reduces to the vacuum equations.

Einstein at one time considered a slightly more general form. That is

$$R_{\alpha\beta} - \tfrac{1}{2} R g_{\alpha\beta} + \Lambda g_{\alpha\beta} = 8\pi T_{\alpha\beta} \tag{7.81}$$

where Λ is known as the *cosmological constant.* From time to time, observations have been made that suggest that the cosmological constant is not zero but that it is the subject of an ongoing debate.

The format of Eq. (7.80) or (7.81) has a quite different appearance than Eq. (7.73). This suggests that the gravitational field is not a Yang–Mills field. But let us look at the situation more closely.

First of all the Fock–Ivanenko 2-vectors were introduced by Fock and Ivanenko to make Dirac's equation compatible with the dictates of general relativity. In this context, they may be regarded as components of a gauge field. (See Problem 8.4)

Secondly, for a coordinate frame, we have

$$\nabla_\alpha M = \frac{\partial}{\partial x^\alpha} M - \mathbf{\Gamma}_\alpha M + M \mathbf{\Gamma}_\alpha \tag{7.82}$$

and

$$\tfrac{1}{2}\mathscr{R}_{\alpha\beta} = \frac{\partial}{\partial x^\alpha}\boldsymbol{\Gamma}_\beta - \frac{\partial}{\partial x^\beta}\boldsymbol{\Gamma}_\alpha - [\boldsymbol{\Gamma}_\alpha, \boldsymbol{\Gamma}_\beta] \tag{7.83}$$

where $\mathscr{R}_{\alpha\beta} = \frac{1}{2}R_{\alpha\beta\eta\nu}\gamma^{\eta\nu}$.

Thus we can identify $-\boldsymbol{\Gamma}_\alpha$ with iqA_α in Eq. (7.66) and $\frac{1}{2}\mathscr{R}_{\alpha\beta}$ with $-iqF_{\alpha\beta}$ in Eq. (7.65). From Eq. (7.67), we then have

$$\nabla_\alpha\mathscr{R}_{\mu\nu} + \nabla_\mu\mathscr{R}_{\nu\alpha} + \nabla_\nu\mathscr{R}_{\alpha\mu} = 0.$$

It is interesting to note that this corresponds to the usual Bianchi identity in general relativity. It is not too difficult to show that

$$\nabla_\alpha\mathscr{R}_{\mu\nu} = \tfrac{1}{2}R_{\mu\nu\theta\phi;\alpha}\gamma^{\theta\phi} + \tfrac{1}{2}(R_{\eta\nu\theta\phi}\Gamma_{\alpha\mu}{}^{\eta} + R_{\mu\eta\theta\phi}\Gamma_{\alpha\nu}{}^{\eta})\gamma^{\theta\phi}. \tag{7.84}$$

If we sum over a cyclic permutation to the indices η, ν, and α, we find that the terms generated by the second term in Eq. (7.84) cancel out. Thus

$$\begin{aligned}\nabla_\alpha\mathscr{R}_{\mu\nu} + \nabla_\mu\mathscr{R}_{\nu\alpha} + \nabla_\nu\mathscr{R}_{\alpha\mu} &= \tfrac{1}{2}(R_{\mu\nu\theta\phi;\alpha} + R_{\nu\alpha\theta\phi;\mu} + R_{\alpha\mu\theta\phi;\nu})\gamma^{\theta\phi} \\ &= 0.\end{aligned}$$

Formally we could derive an equation of the form

$$\frac{1}{\sqrt{-g}}\nabla_\alpha(\sqrt{-g}\tfrac{1}{2}\mathscr{R}^{\theta\alpha}) = \mathscr{J}^\theta$$

by a variation of $\boldsymbol{\Gamma}_\theta$ in the action

$$S = \int(-\tfrac{1}{4}(\tfrac{1}{2}\mathscr{R}^{\alpha\beta}\tfrac{1}{2}\mathscr{R}_{\alpha\beta}) + \mathscr{J}^\alpha\boldsymbol{\Gamma}_\alpha)_0\sqrt{-g}\,\mathrm{d}x^0\,\mathrm{d}x^1\,\mathrm{d}x^2\,\mathrm{d}x^3. \tag{7.85}$$

However, in the context of general relativity the independent field is considered to be the metric tensor $g_{\alpha\beta}$ and not the gauge field $\boldsymbol{\Gamma}_\alpha$. It is true that $\boldsymbol{\Gamma}_\alpha$ depends not only on the metric tensor $g_{\alpha\beta}$ but also on the choice of underlying orthonormal frame (Vierbein). On the other hand the curvature 2-form and therefore the action of Eq. (7.85) depends only on the metric tensor. Thus it is not clear to me that it makes sense to carry out a variation of this action with respect to $\boldsymbol{\Gamma}_\alpha$.

Perhaps one could consider the possibility of permitting generalized versions of $\boldsymbol{\Gamma}_\alpha$ and then carrying out a variation of the action with respect to $\boldsymbol{\Gamma}_\alpha$ and $g_{\alpha\beta}$ independently. An approach similar to this has been applied to a somewhat different action. A discussion of this procedure for the *Palatini action* is carried out in Robert M. Wald's *General Relativity* (1984, pp. 450–459). (If you refer to Wald's text, you should be cautioned that the

operator ∇_α that Wald uses is quite different from the version used in this text.)

The nature of these relationships or nonrelationships leaves the status of gravity as a Yang–Mills field subject to debate. Nonetheless we can show how Einstein's equations (or a slight generalization) can be cast in the form of Eq. (7.73). We first note that the derivation of Eq. (7.78) applies to $\frac{1}{2}\mathscr{R}^{\alpha\beta}$. This implies that

$$\frac{1}{\sqrt{-g}}\nabla_\alpha\left(\sqrt{-g}\left(\frac{1}{\sqrt{-g}}\nabla_\beta(\tfrac{1}{2}f\mathscr{R}^{\alpha\beta})\right)\right) = 0$$

which means that if we define

$$\mathscr{J}^\alpha = \frac{1}{\sqrt{-g}}\nabla_\beta(\tfrac{1}{2}f\mathscr{R}^{\alpha\beta})$$

then $\mathscr{J}^\alpha$ may be considered a "conserved current" in the sense that

$$\frac{1}{\sqrt{-g}}\nabla_\alpha(\sqrt{-g}\,\mathscr{J}^\alpha) = 0.$$

Consider

$$\begin{aligned}\nabla_\beta(\tfrac{1}{2}f\mathscr{R}^{\alpha\beta}) &= \tfrac{1}{4}\nabla_\beta(fR^{\alpha\beta}{}_{\eta\nu}\gamma^{\eta\nu})\\ &= \frac{1}{4}\left(\frac{\partial f}{\partial x^\beta}R^{\alpha\beta}{}_{\eta\nu}\gamma^{\eta\nu} + fR^{\alpha\beta}{}_{\eta\nu,\beta}\gamma^{\eta\nu} - fR^{\alpha\beta}{}_{\eta\nu}\Gamma_{\beta\theta}{}^{\eta}\gamma^{\theta\nu} - fR^{\alpha\beta}{}_{\eta\nu}\Gamma_{\beta\theta}{}^{\nu}\gamma^{\eta\theta}\right)\\ &= \frac{1}{4}\left(\frac{\partial f}{\partial x^\beta}R^{\alpha\beta}{}_{\eta\nu} + fR^{\alpha\beta}{}_{\eta\nu,\beta} - fR^{\alpha\beta}{}_{\theta\nu}\Gamma_{\beta\eta}{}^{\theta} - fR^{\alpha\beta}{}_{\eta\theta}\Gamma_{\beta\nu}{}^{\theta}\right)\gamma^{\eta\nu}.\end{aligned}$$

Since

$$R^{\alpha\beta}{}_{\eta\nu;\beta} = R^{\alpha\beta}{}_{\eta\nu,\beta} + R^{\theta\beta}{}_{\eta\nu}\Gamma_{\beta\theta}{}^{\alpha} + R^{\alpha\theta}{}_{\eta\nu}\Gamma_{\beta\theta}{}^{\beta} - R^{\alpha\beta}{}_{\theta\nu}\Gamma_{\beta\eta}{}^{\theta} - R^{\alpha\beta}{}_{\eta\theta}\Gamma_{\beta\nu}{}^{\theta},$$

we can write

$$\begin{aligned}\nabla_\beta(\tfrac{1}{2}f\mathscr{R}^{\alpha\beta}) = \frac{1}{4}\Bigg(&\frac{\partial f}{\partial x^\beta}R^{\alpha\beta}{}_{\eta\nu} + fR^{\alpha\beta}{}_{\eta\nu;\beta}\\ &- fR^{\theta\beta}{}_{\eta\nu}\Gamma_{\beta\theta}{}^{\alpha} - fR^{\alpha\theta}{}_{\eta\nu}\Gamma_{\beta\theta}{}^{\beta}\Bigg)\gamma^{\eta\nu}\end{aligned} \tag{7.86}$$

Because of the symmetries of the dummy indices

$$R^{\theta\beta}{}_{\eta\nu}\Gamma_{\beta\theta}{}^{\alpha} = 0.$$

Furthermore

$$\Gamma_{\beta\theta}{}^{\beta} = \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\theta} \sqrt{-g}.$$

Thus Eq. (7.86) becomes

$$\nabla_\beta(\tfrac{1}{2} f \mathscr{R}^{\alpha\beta}) = \frac{1}{4}\left(f R^{\alpha\beta}{}_{\eta\nu;\beta} + \left(\frac{\partial f}{\partial x^\beta} - \frac{f}{\sqrt{-g}} \frac{\partial}{\partial x^\beta} \sqrt{-g} \right) R^{\alpha\beta}{}_{\eta\nu} \right) \gamma^{\eta\nu}. \tag{7.87}$$

Because of the Bianchi identity

$$\begin{aligned} R^{\alpha\beta}{}_{\eta\nu,\beta} &= -R^{\alpha\beta}{}_{\nu\beta;\eta} - R^{\alpha\beta}{}_{\beta\eta;\nu} \\ &= -R^{\alpha}{}_{\nu;\eta} + R^{\alpha}{}_{\eta;\nu}. \end{aligned}$$

Thus $R^{\alpha\beta}{}_{\eta\nu;\beta} = 0$ when $R^{\alpha}{}_{\beta} = 0$.

Therefore we can require our "current" to be zero for the vacuum state if we set $f = \sqrt{-g}$. With these results, we have

$$\mathscr{J}^\alpha = \frac{1}{\sqrt{-g}} \nabla_\beta(\tfrac{1}{2}\sqrt{-g}\mathscr{R}^{\alpha\beta}) = \tfrac{1}{2} R^{\alpha}{}_{\eta;\nu}\gamma^{\eta\nu}. \tag{7.88}$$

From Eq. (7.81), we have

$$R^{\alpha}{}_{\alpha} - \tfrac{1}{2} R \delta^{\alpha}_{\alpha} + \Lambda \delta^{\alpha}_{\alpha} = 8\pi T^{\alpha}{}_{\alpha}$$

and thus

$$R - 2R + 4\Lambda = 8\pi T$$

or

$$R = 4\Lambda - 8\pi T$$

where $T = T^{\alpha}{}_{\alpha}$. Thus

$$R^{\alpha}{}_{\eta} = \Lambda \delta^{\alpha}_{\beta} + 8\pi(T^{\alpha}{}_{\eta} - \tfrac{1}{2} T \delta^{\alpha}_{\eta}).$$

Therefore

$$\begin{aligned} \mathscr{J}^\alpha &= \frac{1}{\sqrt{-g}} \nabla_\beta(\tfrac{1}{2}\sqrt{-g}\mathscr{R}^{\alpha\beta}) \\ &= \tfrac{1}{4} R^{\alpha\beta}{}_{\eta\nu;\beta}\gamma^{\eta\nu} \\ &= \tfrac{1}{2} \mathscr{R}^{\alpha\beta}{}_{;\beta} \\ &= \tfrac{1}{2} R^{\alpha}{}_{\eta;\nu}\gamma^{\eta\nu} \\ &= 4\pi(T^{\alpha}{}_{\eta} - \tfrac{1}{2} T\delta^{\alpha}_{\eta})_{;\nu}\gamma^{\eta\nu}. \end{aligned}$$

To summarize:

$$\nabla_\alpha[\quad] = \frac{\partial}{\partial x^\alpha}[\quad] - \mathbf{\Gamma}_\alpha[\quad] + [\quad]\mathbf{\Gamma}_\alpha \tag{7.89}$$

where [ ] represents any Clifford number;

$$\tfrac{1}{2}\mathcal{R}_{\alpha\beta} = \frac{\partial}{\partial x^\alpha}\mathbf{\Gamma}_\beta - \frac{\partial}{\partial x^\beta}\mathbf{\Gamma}_\alpha - [\mathbf{\Gamma}_\alpha, \mathbf{\Gamma}_\beta] \tag{7.90}$$

$$\nabla_\alpha \mathcal{R}_{\eta\nu} + \nabla_\eta \mathcal{R}_{\nu\alpha} + \nabla_\nu \mathcal{R}_{\alpha\nu} = 0 \tag{7.91}$$

$$\begin{aligned}
\frac{1}{\sqrt{-g}}\nabla_\theta(\sqrt{-g}\tfrac{1}{2}\mathcal{R}^{\alpha\theta}) &= \tfrac{1}{4}R^{\alpha\beta}{}_{\eta\nu;\beta}\gamma^{\eta\nu} \\
&= \tfrac{1}{2}\mathcal{R}^{\alpha\beta}{}_{;\beta} \\
&= \tfrac{1}{2}R^{\alpha}{}_{\eta;\nu}\gamma^{\eta\nu} \\
&= \mathcal{J}^\alpha \\
&= 4\pi(T^\alpha{}_\eta - \tfrac{1}{2}T\delta^\alpha_\eta)_{;\nu}\gamma^{\eta\nu}
\end{aligned} \tag{7.92}$$

and

$$\mathcal{J}^\alpha{}_{;\alpha} = \frac{1}{\sqrt{-g}}\nabla_\alpha(\sqrt{-g}\mathcal{J}^\alpha) = 0. \tag{7.93}$$

These last five equations are all characteristic of a Yang–Mills theory. Equation (7.92) in the form

$$\tfrac{1}{2}R^\alpha{}_{\eta;\nu}\gamma^{\eta\nu} = 4\pi(T^\alpha{}_\eta - \tfrac{1}{2}T\delta^\alpha_\eta)_{;\nu}\gamma^{\eta\nu} \tag{7.94}$$

may be considered a slightly generalized form of Einstein's equations with the possibility of allowing a nonzero cosmological constant.

Generally we solve Einstein's equations for the metric tensor. However, it is conceivable to this author that one could solve Einstein's equations for the Fock–Ivanenko coefficients and then use Eq. (5.42) to solve for the metric tensor. In practice this probably would not be an easier method to find solutions. However, the theoretical possibility of such a route would lend credence to the suggestion that Einstein's theory of gravitation can be considered to be a Yang–Mills theory.

Problem 7.12. Assume

$$\nabla_\eta \psi = \left(\frac{\partial}{\partial x^\eta} + \mathrm{i}qA_\eta\right)\psi$$

$$\psi' = M\psi$$

$$\nabla_\eta \psi' = \left(\frac{\partial}{\partial x^\eta} + \mathrm{i}qA_\eta\right)\psi'$$

$$\nabla_\eta M\psi = (\nabla_\eta M)\psi + M\nabla_\eta\psi.$$

Show

$$\nabla_\eta M = \frac{\partial}{\partial x^\eta} M + \mathrm{i}q[A_\eta, M].$$

Problem 7.13. Confirm Eq. (7.77).

Problem 7.14. Show $R_{;\theta} = 2R^{\beta}{}_{\theta,\beta}$. Hint: apply the Bianchi identity to $R^{\alpha\beta}{}_{\eta\nu;\theta}$ and then contract the result on two indices. (This implies that the energy–momentum tensor in Eq. (7.80) must satisfy the relation $T^{\alpha\beta}{}_{;\beta} = 0$.)

7.4 The Migma Chamber of Bogdan Maglich

The intent of this section is to walk the reader through an exercise which demonstrates both the advantages of using a coordinate system with the symmetries of the problem and the pitfalls of using a nonCartesian coordinate system. There is no Clifford algebra contained in this section.

One scheme to produce fusion energy has been advocated by Prof. Bogdan Maglich (Macek and Maglić 1970; Maglić *et al.* 1971; Maglich 1973; Gordon and Johnson 1974). In this scheme deuterium or some other appropriate kind of ions are set on self-colliding orbits confined by a cylindrically symmetric magnetic field. Any student who has studied one year of college physics knows that for a cyclotron where $B_x = B_y = 0$ and B_z is constant, charged particles have circular orbits if they are set in motion in the xy-plane.

In the median plane of Prof. Maglich's migma chamber near the axis of symmetry, $B_z = B_0(1 - k\rho^2)$ where $\rho^2 = x^2 + y^2$, $k > 0$, $(1 - k\rho^2) > 0$, and it is hoped the ions will be confined to a region where this is a good approximation. In such a field, the orbit of a charged particle will have a shorter radius of curvature near the axis of symmetry where the magnetic field is strongest and a longer radius of curvature away from the axis of symmetry where the magnetic field is weaker. For this reason, the orbits are no longer closed paths. Thus a crucial question arises. Will a charged particle, which passes through the axis of symmetry once, repeatedly return to this point where it has a chance to collide with other particles following similar

paths or will it eventually spiral out and away from the axis of symmetry? This problem can be dealt with in cylindrical coordinates. We note that

$$E_k = F_{0k} = \frac{\partial A_k}{c\,\partial t} - \frac{\partial A_0}{\partial x^k} = 0 \quad \text{for } x^k = \rho,\ \theta, \text{ and } z.$$

$$B_\rho = -F_{\theta z} = -\frac{\partial A_z}{\partial \theta} + \frac{\partial A_\theta}{\partial z}, \tag{7.95}$$

$$B_\theta = -F_{z\rho} = -\frac{\partial A_\rho}{\partial z} + \frac{\partial A_z}{\partial \theta}, \tag{7.96}$$

$$B_z = -F_{\rho\theta} = -\frac{\partial A_\theta}{\partial \rho} + \frac{\partial A_\rho}{\partial \theta}. \tag{7.97}$$

Since the electric field $\vec{E} = 0$, there is no need to introduce a non-zero A_0 nor a time dependence in A_ρ, A_θ, or A_z. Because of the cylindrical symmetry, there is no reason to introduce any θ dependency in the vector potential. Furthermore, because of the same symmetry, $B_\theta = 0$. Thus Eqs. (7.95)–(7.97) becomes

$$B_\rho = \frac{\partial A_\theta}{\partial z}, \tag{7.98}$$

$$B_\theta = -\frac{\partial A_\rho}{\partial z} = 0,$$

$$B_z = -\frac{\partial A_\theta}{\partial \rho}. \tag{7.99}$$

Thus, the only component of the vector potential $\mathscr{A}$ which is necessarily nonzero is $A_\theta(\rho, z)$. To solve Eq. (7.99), it might be thought that B_z should be the same in cylindrical coordinates as it is in Cartesian coordinates. However, "B_z" is a very poor notation for what is actually a component of the second-rank Faraday tensor. In Cartesian coordinates for our problem, $F_{10} = F_{20} = F_{30} = F_{23} = F_{31} = 0$, and $F_{12} = F_{xy} = -F_{yx} = -B_0(1 - k\rho^2)$. (Actually for $z \neq 0$, neither F_{23} nor F_{31} would be zero.) At any rate, in the xy-plane

$$\begin{aligned} F_{\rho\theta} &= F_{xy}\frac{\partial x}{\partial \rho}\frac{\partial y}{\partial \theta} + F_{yx}\frac{\partial y}{\partial \rho}\frac{\partial x}{\partial \theta} \\ &= F_{xy}(\cos\theta)(\rho\cos\theta) + F_{yx}(\sin\theta)(-\rho\sin\theta), \end{aligned}$$

or

$$F_{\rho\theta} = \rho F_{xy} = -B_0(\rho - k\rho^3). \tag{7.100}$$

We also know that

$$F_{\rho\theta} = \frac{\partial}{\partial \rho} A_\theta,$$

so

$$A_\theta = -B_0\left(\frac{\rho^2}{2} - \frac{k\rho^4}{4}\right) = \frac{B_0\rho^2}{2}\left(1 - \frac{k\rho^2}{2}\right). \tag{7.101}$$

Actually to guarantee confinement in the z-direction, one should assume some z dependence in the function A_θ. At any rate, for the cylindrically symmetric field,

$$L = \tfrac{1}{2}mc^2[c^2(\dot{t})^2 - (\dot{\rho})^2 - (\rho\dot{\theta})^2 - (\dot{z})^2] + q\dot{\theta}A_\theta(\rho, z). \tag{7.102}$$

Since L does not explicitly depend on t or θ, two constants of motion are

$$\frac{\partial L}{c\,\partial \dot{t}} = mc^2(c\dot{t}) = p_t c^2 \tag{7.103}$$

and

$$\frac{\partial L}{\partial \dot{\theta}} = -mc^2\rho^2\dot{\theta} + qA_\theta = -p_\theta c^2. \tag{7.104}$$

We also know that since

$$(\mathrm{d}s)^2 = c^2(\mathrm{d}t)^2 - (\mathrm{d}\rho)^2 - \rho^2(\mathrm{d}\theta)^2 - (\mathrm{d}z)^2,$$

it follows that

$$c^2(\dot{t})^2 - \dot{\rho}^2 - \rho^2\dot{\theta}^2 - \dot{z}^2 = 1. \tag{7.105}$$

Combining Eqs. (7.103)–(7.105), we get

$$\dot{\rho}^2 + \dot{z}^2 = \frac{A^2\rho^2 - (c^2p_\theta + qA_\theta)^2}{(mc^2)^2\rho^2}, \tag{7.106}$$

where

$$A^2 = ((p_t)^2c^4 - m^2c^4) = (mc^2)^2u^2 = (mcv)^2/(1 - (v/c)^2). \tag{7.107}$$

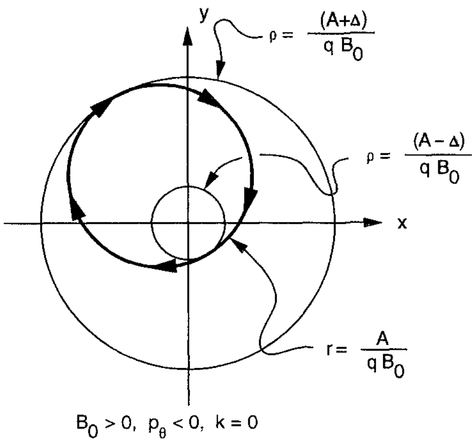


Fig. 7.2. Circular orbit of a positive ion in a uniform magnetic field when the constant generalized angular momentum $p_\theta = m\rho^2\dot{\theta} - qA_\theta/c^2$ is negative.

In the case for which $\dot{z} = 0$ and $k = 0$, Eqs. (7.104) and (7.106) become

$$\dot{\theta} = \frac{p_\theta}{m\rho^2} - \frac{qB_0}{2mc^2} \tag{7.108}$$

and

$$(\dot{\rho})^2 = \frac{A^2\rho^2 - (c^2 p_\theta + qA_\theta)^2}{(mc^2)\rho^2}. \tag{7.109}$$

If $B_0 > 0$, we have three cases. (See Figs. 7.2, 7.3, and 7.4.)

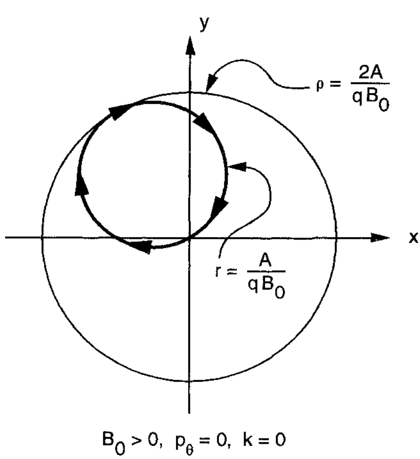


Fig. 7.3. Circular orbit of a positive ion in a uniform magnetic field when the generalized angular momentum p_θ is zero.

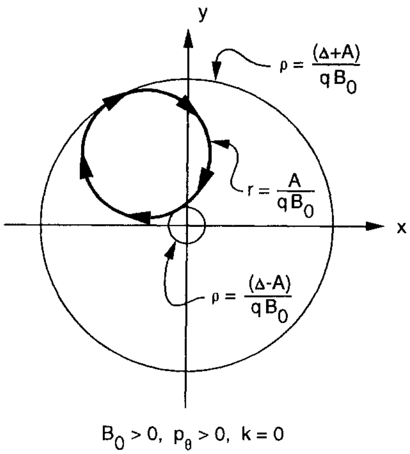


Fig. 7.4. Circular orbit of a positive ion in a uniform magnetic field when the generalized angular momentum p_θ is positive.

If $p_\theta < 0$, then from Eq. (7.108), $\dot{\theta}$ remains negative throughout the orbit. The coordinate ρ has two critical points when $\dot{\rho} = 0$. At all points in the orbit, the magnetic field has the same sign so the curvature never reverses sign. This means that at these critical points $\dot{\rho}$ reverses its sign and ρ attains a maximum or minimum value. According to Eq. (7.109), there are four values of ρ which make $\dot{\rho} = 0$. But two of them are negative and have no physical meaning. The two remaining roots are

$$\rho_{\text{MAX}} = \frac{(A + \Delta)}{qB_0} \qquad \text{and} \qquad \rho_{\text{MIN}} = \frac{(A - \Delta)}{qB_0}$$

where

$$\Delta = \sqrt{A^2 + 2qB_0 p_\theta c^2}. \tag{7.110}$$

(See Fig. 7.2.)

If $p_\theta = 0$, $\dot{\theta}$ again remains negative throughout the orbit. In this case

$$(mc^2)^2(\dot{\rho})^2 = A^2 - \frac{q^2(B_0)^2}{4}\rho^2.$$

The value of $\dot{\rho}$ is zero at only one point of the orbit. This occurs when

$$\rho = \rho_{\text{MAX}} = \frac{2A}{qB_0}.$$

The minimum value of ρ is 0. At that point $\dot{\rho}$ is discontinuous. In particular,

the value of $\dot{\rho}$ jumps in value from $-A/mc^2$ to $+A/mc^2$ as the particle passes through the origin. (See Fig. 7.3.)

If $p_\theta > 0$, then from Eq. (7.108), $\dot{\theta} < 0$ for

$$\rho > \sqrt{\frac{2p_\theta c^2}{B_0}}$$

while $\dot{\theta} > 0$ for

$$\rho < \sqrt{\frac{2p_\theta c^2}{B_0}}.$$

As in the first case, the coordinate ρ has two turning points where $\dot{\rho} = 0$. They are

$$\rho_{\text{MAX}} = \frac{(\Delta + A)}{qB_0} \qquad \text{and} \qquad \rho_{\text{MIN}} = \frac{(\Delta - a)}{qB_0},$$

where Δ is defined by Eq. (7.110).

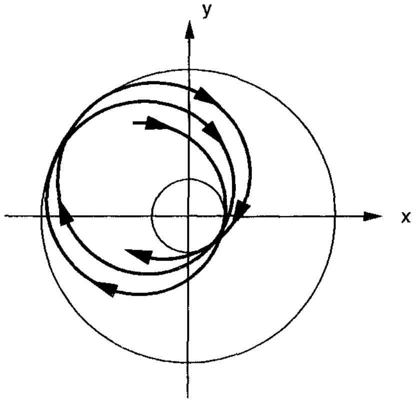
In all three cases, the orbit of the particle is a circle with radius

$$r = \frac{A}{qB_0} = \frac{mcv}{qB_0\sqrt{1-(v/c)^2}}. \tag{7.111}$$

The choice of the cylindrical coordinate system would be a very poor choice if our goal was to show that these orbits are circles. This can be done far more easily in Cartesian coordinates. (See Problem 7.15.) The advantage of using the cylindrical coordinate system is that it is now possible to see what happens when B_z (in Cartesian coordinates) is $B_0(1 - k\rho^2)$ instead of B_0. In that case A_θ is $-\frac{1}{2}B_0\rho^2(1 - \frac{1}{2}k\rho^2)$ instead of $-\frac{1}{2}B_0\rho^2$. This has the effect of replacing B_0 in Eqs. (7.108) and (7.109) by $B_0(1 - \frac{1}{2}k\rho^2)$.

Since k is small, this causes only a slight shift in the turning points of ρ in each of the three cases discussed above. It is still necessary for p_θ to be zero in order for the particle to pass through the origin.

The most significant change is that the paths are no longer circles. Since the radius of curvature is inversely proportional to the strength of the magnetic field, the path of a particle will straighten out somewhat when the particle recedes from the axis of symmetry. Thus a sort of precession of the orbit will occur. (See Figs. 7.5, 7.6, and 7.7.) Although the orbits are no longer circles, it is clear that these orbits remain bounded. Furthermore, because of the precession of these orbits, particles with positive values of p_θ will tend to collide head-on with particles with negative values of p_θ. Therefore if a large number of ions can be set in orbits that pass near the



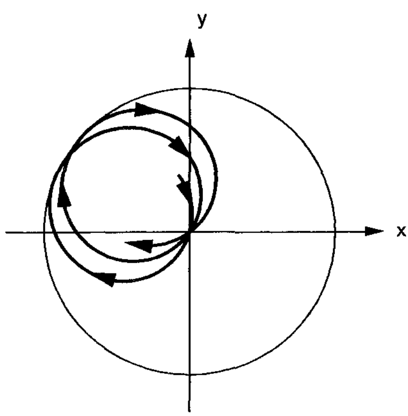
$B_0 > 0,\ p_\theta < 0,\ k > 0$

Fig. 7.5. Self-colliding orbit of positive ion in a nonuniform Maglich magnetic field when the generalized momentum p_θ is negative.

axis of symmetry, one can reasonably expect to get some collisions and fusion events.

Problem 7.15. It is shown in virtually every Freshman level college physics course that without relativistic considerations the orbits of ions in planes perpendicular to a uniform magnetic field are circles. It can be shown that this fact remains true according to the laws of special relativity.

Suppose $\vec{E} = 0$ and $\vec{B} = (B_x, B_y, B_z) = (0, 0, B_0)$. Consider a charged



$B_0 > 0,\ p_\theta = 0,\ k > 0$

Fig. 7.6. Self-colliding orbit of positive ion in a nonuniform Maglich magnetic field when the generalized momentum p_θ is zero.

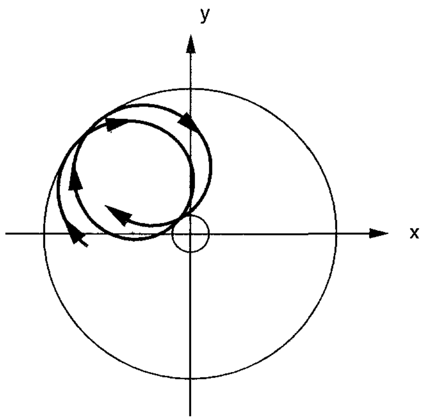


Fig. 7.7. Self-colliding orbit of positive ion in a nonuniform Maglich magnetic field when the generalized momentum p_θ is positive.

particle with initial world velocity $\boldsymbol{u}(0) = (u^0(0), u^1(0), u^2(0), 0)$, where

$$u^0(0) = \frac{1}{\sqrt{1-(v/c)^2}}, \qquad u^1(0) = \frac{v_x(0)}{c\sqrt{1-(v/c)^2}},$$

and

$$u^2(0) = \frac{v_y(0)}{c\sqrt{1-(v/c)^2}}.$$

(1) Use the result of part (4) of Problem 7.8 to compute $\boldsymbol{u}(s)$.
(2) Integrate both sides of the equation $\dot{\boldsymbol{x}}(s) = \boldsymbol{u}(s)$ to obtain a formula for $\boldsymbol{x}(s)$ in terms of $\boldsymbol{x}(0)$ and $\boldsymbol{u}(0)$.
(3) Show that

$$(x^1(s) - a)^2 + (x^2(s) - b)^2 = \frac{(mcv)^2}{(qB_0)^2(1-(v/c)^2)},$$

and

$$a = x^1(0) + \frac{mc^2}{qB_0}u^2(0), \qquad \text{and} \qquad b = x^2(0) - \frac{mc^2}{qB_0}u^1(0).$$

7.5 The Generalized Stoke's Theorem

In this section I will use both the terminology and notation of differential forms. In Eq. 7.19, it was observed that an upper index coordinate Dirac

matrix arises when we apply the exterior derivative $\mathbf{d}$, to a coordinate. That is

$$\gamma^\alpha = \gamma^\beta \wedge \nabla_\beta x^\alpha = \mathbf{d}x^\alpha. \tag{7.112}$$

In the formalism of differential forms, $\mathbf{d}x^\alpha$ is a *1-form* and

$$\begin{aligned}\mathscr{G} &= \frac{1}{p!} G_{\alpha_1\alpha_2\ \ .\alpha_p} \mathbf{d}x^{\alpha_1} \wedge \mathbf{d}x^{\alpha_2} \wedge \ldots \wedge \mathbf{d}x^{\alpha_p} \\ &= \frac{1}{p!} G_{\alpha_1\alpha_2\ldots\alpha_p} \mathbf{d}x^{\alpha_1}\, \mathbf{d}x^{\alpha_2} \ldots \mathbf{d}x^{\alpha_p}\end{aligned}$$

is a *p-form.*

In the formalism of differential forms, there is a meaningful distinction between p-vectors and p-forms. However, this distinction will not have any impact on the particular computations carried out in this section.

The *generalized Stoke's theorem* is

$$\int_V \mathbf{d}\mathscr{F} = \int_{\partial V} \mathscr{F} \tag{7.113}$$

where $\mathscr{F}$ is a $(p-1)$-form, $\mathbf{d}\mathscr{F}$ is a p-form, and ∂V is a $(p-1)$-dimensional boundary of the p-dimensional volume V. For $p = 2$, this is the usual Stoke's theorem. For $p = 3$, this is generally known as Gauss's theorem. Essentially what the theorem says is that in any dimension there are special cases for which one can convert a volume integral to a surface integral.

Because a surface has an orientation, some care must be taken in the evaluation of a surface integral. This section is devoted to proving the generalized Stoke's theorem for volumes that have shapes that are easy to deal with. In the process of carrying out the proof it is also our hope to make the surface integral on the right-hand side of Eq. (7.113) computationally meaningful.

To underline possible pitfalls, let us consider an example of what the theorem does not say. Suppose

$$\mathscr{F} = F_1\, \mathbf{d}x^1 + F_2\, \mathbf{d}x^2. \tag{7.114}$$

Then

$$\mathbf{d}\mathscr{F} = \frac{\partial F_2}{\partial x^1}\mathbf{d}x^1\, \mathbf{d}x^2 + \frac{\partial F_1}{\partial x^2}\mathbf{d}x^2\, \mathbf{d}x^1. \tag{7.115}$$

So far, so good.

Now let us try to compute $\int_V \mathbf{d}\mathscr{F}$ where V is the 2-dimensional region

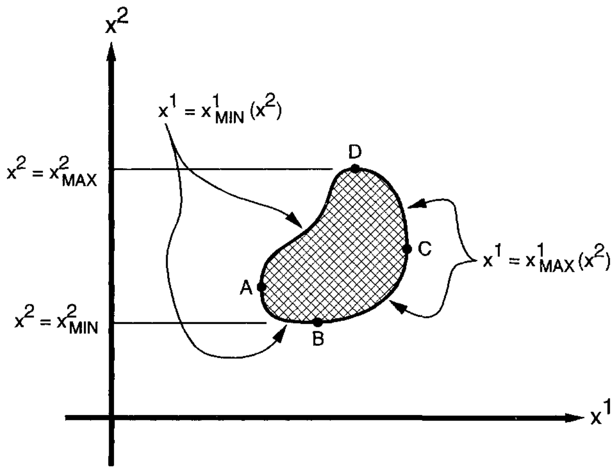


Fig. 7.8. The path BCD is defined by the equation $x^1 = x^1_{\text{MAX}}(x^2)$ while the path BAD is defined by the equation $x^1 = x^1_{\text{MIN}}(x^2)$.

shown in Figs. 7.8 and 7.9. Our initial inclination might be to write

$$\begin{aligned}\int_V \mathrm{d}\mathscr{F} &= \int_V \frac{\partial F_2}{\partial x^1}\,\mathrm{d}x^1\,\mathrm{d}x^2 + \int_V \frac{\partial F_1}{\partial x^2}\,\mathrm{d}x^2\,\mathrm{d}x^1 \\ &= \int_{x^2_{\text{MIN}}}^{x^2_{\text{MAX}}} \left(\int_{x^1_{\text{MIN}}(x^2)}^{x^1_{\text{MAX}}(x^2)} \frac{\partial F_2(x^1, x^2)}{\partial x^1}\,\mathrm{d}x^1 \right) \mathrm{d}x^2 \\ &\quad + \int_{x^1_{\text{MIN}}}^{x^1_{\text{MAX}}} \left(\int_{x^2_{\text{MIN}}(x^1)}^{x^2_{\text{MAX}}(x^1)} \frac{\partial F_1(x^1, x^2)}{\partial x^2}\,\mathrm{d}x^2 \right) \mathrm{d}x^1,\end{aligned}$$

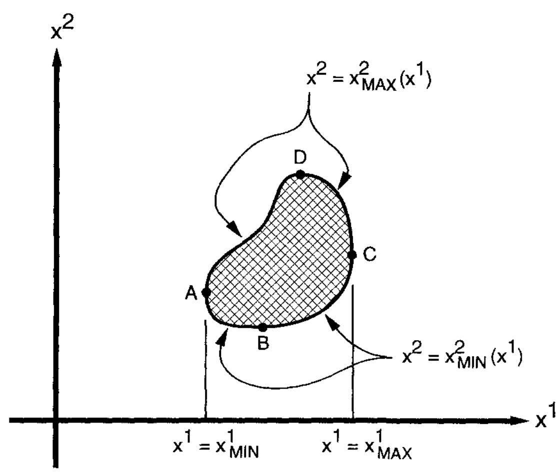


Fig. 7.9. The path ABC is defined by the equation $x^2 = x^2_{\text{MIN}}(x^1)$ while the path ADC is defined by the equation $x^2 = x^2_{\text{MAX}}(x^1)$.

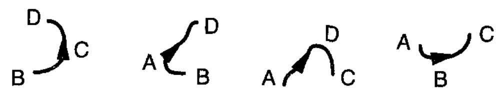
or

$$\int_V \mathbf{d}\mathscr{F} = \int_{x^2_{\text{MIN}}}^{x^2_{\text{MAX}}} F_2(x^1_{\text{MAX}}(x^2), x^2)\,\mathrm{d}x - \int_{x^2_{\text{MIN}}}^{x^2_{\text{MAX}}} F_2(x^1_{\text{MIN}}(x^2), x^2)\,\mathrm{d}x^2$$
$$+ \int_{x^1_{\text{MIN}}}^{x^1_{\text{MAX}}} F_1(x^1, x^2_{\text{MAX}}(x^1)\,\mathrm{d}x^1 - \int_{x^1_{\text{MIN}}}^{x^1_{\text{MAX}}} F_1(x^1, x^2_{\text{MIN}}(x^1)\,\mathrm{d}x^1. \quad (7.116)$$

(You should refer to Fig. 7.8 to obtain an understanding of the notation used for the first two integrals on the right-hand side of Eq. (7.116). You should also refer to Fig. 7.9 to get a handle on the notation used for the last two integrals of the right-hand side of Eq. (7.116).)

What we now have is four path integrals:

$$\int_V \mathbf{d}\mathscr{F} = \int F_2\,\mathrm{d}x^2 - \int F_2\,\mathrm{d}x^2 + \int F_1\,\mathrm{d}x^1 - \int F_1\,\mathrm{d}x^1,$$



that is

$$\int \mathbf{d}\mathscr{F} = \int F_2\,\mathrm{d}x^2 + \int F_1\,\mathrm{d}x^1. \quad (7.117)$$

Something here is clearly amiss! On the right-hand side of Eq. (7.117), we have one counterclockwise path integral and one clockwise path integral. This does not look like Stoke's theorem. The usual Stoke's theorem tells us that

$$\int F_1\,\mathrm{d}x^1 + F_2\,\mathrm{d}x^2 = \int \left(\frac{\partial F_2}{\partial x^1} - \frac{\partial F_1}{\partial x^2}\right) \mathrm{d}x^1\,\mathrm{d}x^2.$$

Thus it appears that we should have first written Eq. (7.115) in the form

$$\mathbf{d}\mathscr{F} = \left(\frac{\partial F_2}{\partial x^1} - \frac{\partial F_1}{\partial x^2}\right) \mathbf{d}x^1\,\mathbf{d}x^2.$$

Then we should formally replace the 1-forms $\mathbf{d}x^1$ and $\mathbf{d}x^2$ by the scalar

differentials $\mathrm{d}x^1$ and $\mathrm{d}x^2$ under the integration symbol. We would then have

$$\int_V \mathbf{d}\mathscr{F} = \int\left(\frac{\partial F_2}{\partial x^1} - \frac{\partial F_1}{\partial x^2}\right)\mathrm{d}x^1\,\mathrm{d}x^2 = \int F_1\,\mathrm{d}x^1 + F_2\,\mathrm{d}x^2 = \int_{\partial V} \mathscr{F}.$$

We now turn to the problem of constructing a set of sign conventions which make the generalized version of Stoke's theorem meaningful. It is generally not too difficult to get a handle on a volume integral if the shape is not too horrendous. However, a boundary integral is another matter. In the example just discussed, our boundary integral was a counterclockwise integral. But what does "counterclockwise" mean in higher dimensions? Most of us have a pretty good idea of the orientation of a 2-dimensional boundary to a 3-dimensional solid. However, beyond three dimensions, my intuition fails—at least in this context.

Our approach in this section is to first codify the meaning of a volume integral. This will then define what is meant by

$$\int_V \mathrm{d}\mathscr{F}.$$

For simply shaped volumes, we will carry out the obvious partial integrations indicated and thereby determine what is really meant by the boundary integral

$$\int_{\partial V} \mathscr{F}.$$

What we then have is not a theorem but a workable definition of a boundary integral.

For the integral of an n-form, there is only one possible definition that is both plausible and simple. Suppose

$$\begin{aligned}\mathscr{F} &= \frac{1}{n!}F_{\alpha_1\alpha_2\,..\,\alpha_n}\,\mathbf{d}x^{\alpha_1}\,\mathbf{d}x^{\alpha_2}\ldots\mathbf{d}x^{\alpha_n}\\ &= F_{12\,..\,n}\,\mathbf{d}x^1\,\mathbf{d}x^2\ldots\mathbf{d}x^n.\end{aligned}$$

To compute the integral of $\mathscr{F}$, one formally replaces the n-form $\mathbf{d}x^1\,\mathbf{d}x^2\ldots\mathbf{d}x^n$ by the product of scalar differentials $\mathrm{d}x^1\,\mathrm{d}x^2\ldots\mathrm{d}x^n$ and

writes

$$\int_V \mathscr{F} = \int_V F_{12\ \ n}\, dx^1\, dx^2 \ldots dx^n$$
$$= \int_{x^n_{\mathrm{MIN}}}^{x^n_{\mathrm{MAX}}} \left(\ldots \int_{x^2_{\mathrm{MIN}}}^{x^2_{\mathrm{MAX}}} \left(\int_{x^1_{\mathrm{MIN}}}^{x^1_{\mathrm{MAX}}} (F_{12\ \ n}\, dx^1)\, dx^2 \right) \ldots \right) dx^n, \quad (7.118)$$

where it is understood that

$$x^k_{\mathrm{MAX(MIN)}} = x^k_{\mathrm{MAX(MIN)}}(x^{k+1}, x^{k+2}, \ldots, x^n) \quad \text{for } k = 1, 2, \ldots, n-1.$$

(To make things easy for ourselves, we will consider only volumes such that we can use a single coordinate system. That is we don't have to patch together different coordinate systems to cover the entire volume. Furthermore, we will assume that the shape of V with respect to our chosen coordinate system is such that if one considers curves obtained by freezing $n - 1$ of the x^k's and letting the remaining coordinate vary, then any one of those curves will either cross the surface of the volume at exactly two points, touch the surface tangently at one point, or miss the volume entirely.)

For a p-form, the situation becomes considerably more complex. Here we would have

$$\int_V \mathscr{F} = \frac{1}{p!} \int_V F_{\alpha_1 \alpha_2 \ldots \alpha_p}\, \mathbf{d}x^{\alpha_1}\, \mathbf{d}x^{\alpha_2} \ldots \mathbf{d}x^{\alpha_p}. \quad (7.119)$$

On the face of it, this could be the sum of $\binom{n}{p}$ distinct integrals. However, we will assume that the p-dimensional volume V can be parameterized with p parameters $u^1, u^2, \ldots, u^p$. That is, for any point in the p-dimensional volume,

$$x^\alpha = x^\alpha(u^1, u^2, \ldots, u_p) \quad \text{for } \alpha = 1, 2, \ldots, n.$$

In this situation,

$$\mathbf{d}x^{\alpha_1}\, \mathbf{d}x^{\alpha_2} \ldots \mathbf{d}x^{\alpha_p} = \frac{\partial x^{\alpha_1}}{\partial u^{\beta_1}} \frac{\partial x^{\alpha_2}}{\partial u^{\beta_2}} \cdots \frac{\partial x^{\alpha_p}}{\partial u^{\beta_p}}\, \mathbf{d}u^{\beta_1}\, \mathbf{d}u^{\beta_2} \ldots \mathbf{d}u^{\beta_p},$$

and Eq. (7.119) becomes

$$\int_V \mathscr{F} = \frac{1}{p!} \int_V F'_{\beta_1 \beta_2 \ldots \beta_p}\, \mathbf{d}u^{\beta_1}\, \mathbf{d}u^{\beta_2} \ldots \mathbf{d}u^{\beta_p}$$
$$= \int_V F'_{12\ldots p}\, du^1\, \mathbf{d}u^2 \ldots \mathbf{d}u^p, \quad (7.120)$$

where

$$F'_{\beta_1\beta_2\dots\beta_p} = F_{\alpha_1\alpha_2\dots\alpha_p} \frac{\partial x^{\alpha_1}}{\partial u^{\beta_1}} \frac{\partial x^{\alpha_2}}{\partial u^{\beta_2}} \cdots \frac{\partial x^{\alpha_p}}{\partial u^{\beta_p}}. \tag{7.121}$$

Dropping the prime, we now define the integral to be the result of formally replacing the p-form $\mathbf{d}u^1\,\mathbf{d}u^2\dots\mathbf{d}u^p$ by the product of scalar differentials $\mathrm{d}u^1\,\mathrm{d}u^2\dots\mathrm{d}u^p$ and then carrying out the indicated integration; that is

$$\begin{aligned}\int_V \mathscr{F} &= \int_V F_{12\dots p}\,\mathrm{d}u^1\,\mathrm{d}u^2\dots\mathrm{d}u^p \\ &= \int_{u^p_{\text{MIN}}}^{u^p_{\text{MAX}}} \left(\cdots \int_{u^2_{\text{MIN}}}^{u^2_{\text{MAX}}} \left(\int_{u^1_{\text{MIN}}}^{u^1_{\text{MAX}}} (F_{12\dots p}\,\mathrm{d}u^1)\,\mathrm{d}u^2\right)\cdots\right)\mathrm{d}u^p, \end{aligned} \tag{7.122}$$

where it is understood that

$$u^{\beta}_{\text{MAX(MIN)}} = u^{\beta}_{\text{MAX(MIN)}}(u^{\beta+1}, u^{\beta+2}, \dots, u^p) \quad \text{for } \beta = 1, 2, \dots, p.$$

Now let us turn to the problem of evaluating the integral $\int_V \mathbf{d}\mathscr{F}$, where V is a p-dimensional volume. Much as we did before, we will assume that the coordinates of the volume and its boundary can be parameterized in terms of p variables. That is, for any point in the volume or on its boundary

$$x^\alpha = x^\alpha(u^1, u^2, \dots, u^p) \quad \text{for } \alpha = 1, 2, \dots, n.$$

Since we are concerned with the $(p-1)$-form $\mathscr{F}$ only on the boundary of V, we can use the same p variables to describe $\mathscr{F}$. That is, we can write

$$\mathscr{F} = \sum_{k=1}^{p} F_{12\dots\hat{k}\dots p}\,\mathbf{d}u^1\,\mathbf{d}u^2\dots\widehat{\mathbf{d}u^k}\dots\mathbf{d}u^p, \tag{7.123}$$

where the caps over the index k and $\mathbf{d}u^k$ indicate missing entities. Using this notation, we have

$$\begin{aligned}\mathbf{d}\mathscr{F} &= \sum_{k=1}^{p} \frac{\partial}{\partial u^k} F_{12\dots\hat{k}\dots p}\,\mathrm{d}u^k\,\mathrm{d}u^1\,\mathrm{d}u^2\dots\widehat{\mathrm{d}u^k}\dots\mathrm{d}u^p \\ &= \sum_{k=1}^{p} (-1)^{k-1} \frac{\partial}{\partial u^k} F_{12\dots\hat{k}\dots p}\,\mathrm{d}u^1\,\mathrm{d}u^2\dots\mathrm{d}u^k\dots\mathrm{d}u^p. \end{aligned} \tag{7.124}$$

Now from our previous definitions, we have

$$\int_V \mathbf{d}\mathscr{F} = \sum_{k=1}^{p} (-1)^{k-1} \int\cdots\int\int \frac{\partial}{\partial u^k} F_{12\dots\hat{k}\dots p}\,\mathrm{d}u^1\,\mathrm{d}u^2\dots\mathrm{d}u^p \tag{7.125}$$

Singling out a particular value of k in the summation in Eq. (7.125) and integrating over u^k, one has

$$\int \cdots \iint \frac{\partial}{\partial u^k} F_{12\ldots\hat{k}\ldots p}\, du^1\, du^2 \ldots du^p$$
$$= \int \cdots \iint [F_{12\ldots\hat{k}\ldots p}(u^1, u^2, \ldots, u^k_{\mathrm{MAX}}, \ldots, u^p)$$
$$- F_{12\ldots\hat{k}\ldots p}(u^1, u^2, \ldots, u^k_{\mathrm{MIN}}, \ldots, u^p)]\, du^1\, du^2 \ldots \widehat{du^k} \ldots du^p,$$

where

$$u^k_{\mathrm{MAX(MIN)}} = u^k_{\mathrm{MAX(MIN)}}(u^1, u^2, \ldots, \widehat{u^k}, \ldots, u^p).$$

Thus if we require that

$$\int_{\partial V} \mathscr{F} = \int_{\partial V} \sum_{k=1}^{p} F_{12\ldots\hat{k}\ldots p}\, \mathbf{d}u^1\, \mathbf{d}u^2 \ldots \widehat{\mathbf{d}u^k} \ldots \mathbf{d}u^p = \int_V \mathbf{d}\mathscr{F},$$

then we are essentially forced to define

$$\int_{\partial V} F_{12\ldots\hat{k}\ldots p}\, du^1\, du^2 \ldots \widehat{du^k} \ldots du^p$$
$$= (-1)^{k-1} \int \cdots \iint [F_{12\ldots\hat{k}\ldots p}(u^1, u^2, \ldots, u^k_{\mathrm{MAX}}, \ldots, u^p)$$
$$- F_{12\ldots\hat{k}\ldots p}(u^1, u^2, \ldots, u^k_{\mathrm{MIN}}, \ldots, u^p)]\, du^1\, du^2 \ldots \widehat{du^k} \ldots du^p. \quad (7.126)$$

This is the main result of this section.

Problem 7.16. Show that Eq. (7.121) can be rewritten in the form

$$F'_{12\ldots p} = \frac{1}{p!} F_{\alpha_1\alpha_2\ldots\alpha_p} \frac{\partial(x^{\alpha_1}, x^{\alpha_2}, \ldots, x^{\alpha_p})}{\partial(u^1, u^2, \ldots, u^p)}.$$

where

$$\frac{\partial(x^{\alpha_1}, x^{\alpha_2}, \ldots, x^{\alpha_p})}{\partial(u^1, u^2, \ldots, u^p)}$$

is the Jacobian of the coordinate transformation.

Problem 7.17. Show that the integral of a p-form as defined by Eq. (7.122) is independent of the choice of variables used to parameterize the volume V.

Problem 7.18. In vector notation, Gauss's theorem is

$$\int_V (\vec{F}\cdot\hat{n})\,\mathrm{d}s = \int_V \vec{\nabla}\cdot\vec{F}\,\mathrm{d}V,$$

where $\hat{n}$ is the unit vector perpendicular to the surface in the outward direction. Show that this theorem is encompassed in the generalized Stoke's theorem if we make the identifications $F_x = F_{23}$, $F_y = -F_{13} = F_{31}$, and $F_z = F_{12}$. (Aside from checking the various sign conventions, one also needs to verify at least informally that

$$\int_V F_x n_x\,\mathrm{d}s = \iint F_x(x_{\mathrm{MAX}}(y,z),y,z)\,\mathrm{d}y\,\mathrm{d}z - \iint F_x(x_{\mathrm{MIN}}(y,z),y,z)\,\mathrm{d}y\,\mathrm{d}z$$

and similar relations for $\int_V F_y n_y\,\mathrm{d}s$ and $\int_V F_z n_z\,\mathrm{d}s$.)

Problem 7.19. Apply the generalized Stoke's theorem twice to show that $\int_{\partial\partial V} \mathscr{F} = 0$. (What does this say about $\partial\partial V$?)

8

DIRAC'S EQUATION FOR THE ELECTRON

8.1 Currents and Dipoles in Curved Space Resulting from Dirac's Equation for the Electron

P. A. M. Dirac (1928) introduced what are now known as Dirac matrices to formulate his equation for the wave function ψ of an electron:

$$\gamma^\eta\left(\frac{\hbar}{\mathrm{i}}\frac{\partial}{\partial x^\eta} + \frac{e}{c}A_\eta\right)\psi = -mc\psi. \tag{8.1}$$

It is understood that $\gamma^\eta A_\eta = \mathscr{A}$ is the 4-vector electromagnetic potential encountered in Section 7.2.

The usual approach is to introduce specific 4×4 matrix representations for the Clifford numbers and then look for four component column vector solutions. The disadvantage to this approach is that the components of these vector wave functions depend on the matrix representation chosen for the Clifford numbers. This means that the components do not lend themselves to physical interpretation.

Soon after Dirac's paper appeared, several physicists presented formulations which did not depend on specific representations of the Dirac matrices. These include Eddington 1928; Proca 1930; Sauter 1930; Franz 1935. In the second volume of the 1939 edition of his text *Atombau und Spektrallinien*, Arnold Sommerfeld devoted over 100 pages to the application of Clifford algebra to the Dirac equation. This is focused on solutions written in terms of Clifford numbers. Nonetheless, this approach did not attract a large group of users at that time.

However, almost 20 years later, when discussing solutions of Dirac's equation for the Coulomb potential, Paul C. Martin and Roy J. Glauber (1958) found that from a computational point of view it is useful to rely on the algebraic properties of the Dirac matrices rather than on any specific matrix representation.

In recent years, physicists have developed a bunch of algebraic tricks and computational shortcuts in the manipulation of Dirac matrices for

Minkowski 4-space. Many of these are summarized in a paper by Alberto Sirlin (1981).

To obtain solutions of Eq. (8.1) in terms of Clifford numbers, it is clear that we need to deal with linear combinations of p-vectors with complex coefficients. This requires some new definitions.

To obtain the *complex conjugate* $\bar{\mathscr{A}}$ of a Clifford number $\mathscr{A}$, we take the complex conjugate of each coefficient of each product of Dirac matrices that appears in the sum representing $\mathscr{A}$. That is, if

$$\mathscr{A} = IA + \gamma_i A^i + \gamma_{ij} A^{ij} + \gamma_{ijk} A^{ijk} + \gamma_{ijkm} A^{ijkm}, \tag{8.2}$$

then

$$\bar{\mathscr{A}} = I\bar{A} + \gamma_i \bar{A}^i + \gamma_{ij} \bar{A}^{ij} + \gamma_{ijk} \bar{A}^{ijk} + \gamma_{ijkm} \bar{A}^{ijkm}, \tag{8.3}$$

where $\bar{A}^{i_1 i_2 \cdots i_n}$ denotes the complex conjugate of $A^{i_1 i_2 \cdots i_n}$.

It is also useful to define the *complex reverse* $\mathscr{A}^\dagger$ of a Clifford number. The complex reverse of a Clifford number $\mathscr{A}$ is obtained by taking the complex conjugate of $\mathscr{A}$ and then reversing the order of all Dirac matrices that appear in a-product. Thus if $\mathscr{A}$ is represented by Eq. (8.2), then

$$\mathscr{A}^\dagger = I\bar{A} + \gamma_i \bar{A}^i + \gamma_{ji} \bar{A}^{ij} + \gamma_{kji} \bar{A}^{ijk} + \gamma_{mkji} \bar{A}^{ijkm}. \tag{8.4}$$

If $\mathscr{A}^\dagger = \mathscr{A}$ then we will say that $\mathscr{A}$ is *c-Hermitian*. If $\mathscr{A}^\dagger = \mathscr{A}^{-1}$ then we will describe $\mathscr{A}$ as being *c-unitary*.

Since the complex reverse reduces to the ordinary reverse for real Clifford numbers we can use the same notation to define a *complex scalar product*. One simply writes

$$\langle \mathscr{A}, \mathscr{B} \rangle = (\mathscr{A}^\dagger \mathscr{B})_0, \tag{8.5}$$

where as before $(\mathscr{A}^\dagger \mathscr{B})_0$ designates the 0-vector component of $\mathscr{A}^\dagger \mathscr{B}$.

Most of the work done on the Dirac equation has been done for the flat space metric of special relativity. However, Vladimir A. Fock and Dimitriĭ Ivanenko (1929) pointed out that Dirac's equation could be adjusted for curved space. They did this by introducing what have since become known as Fock–Ivanenko coefficients. These were discussed in Chapter 5 of this book as Fock–Ivanenko 2-vectors. With the Fock–Ivanenko adjustment, Dirac's equation for curved space becomes

$$\gamma^\eta \left(\frac{\partial}{\partial x^\eta} - \mathbf{\Gamma}_\eta + \frac{ie}{\hbar c} A_\eta \right) \psi = -\frac{imc}{\hbar} \psi. \tag{8.6}$$

From Eq. (5.44), we know that

$$\left(\frac{\partial}{\partial x^\eta} - \Gamma_\eta \right) [\quad] = \nabla_\eta [\quad] - [\quad] \Gamma_\eta \tag{8.7}$$

where any differentiable Clifford number can be placed in the square brackets. Using this equation, one can rewrite Eq. (8.6) in the form

$$\gamma^\eta \nabla_\eta \psi - \gamma^\eta \psi \mathscr{B}_\eta = -\frac{imc}{\hbar}\psi, \tag{8.8}$$

where

$$\mathscr{B}_\eta = \mathbf{\Gamma}_\eta - \frac{ie}{\hbar c} A_\eta I. \tag{8.9}$$

From these equations it is not too difficult to construct a covariant current. If one takes the complex reverse of Eq. (8.6), one gets

$$(\nabla_\eta \psi^\dagger)\gamma^\eta - \mathscr{B}_\eta^\dagger \psi^\dagger \gamma^\eta = \frac{imc}{\hbar}\psi^\dagger. \tag{8.10}$$

Since $\mathbf{\Gamma}_\eta$ is a 2-vector with real coefficients, it is clear from Eq. (8.9) that

$$\mathscr{B}_\eta^\dagger = -\mathscr{B}_\eta. \tag{8.11}$$

If we substitute this into Eq. (8.10), we have

$$(\nabla_\eta \psi^\dagger)\gamma^\eta + \mathscr{B}_\eta \psi^\dagger \gamma^\eta = \frac{imc}{\hbar}\psi^\dagger. \tag{8.12}$$

If we now multiply this equation from the right by ψ and multiply Eq. (8.8) from the left by $\psi^\dagger$ and then sum the two resulting equations, we get

$$(\nabla_\eta \psi^\dagger)\gamma^\eta \psi + \psi^\dagger \gamma^\eta \nabla_\eta \psi + \mathscr{B}_\eta(\psi^\dagger \gamma^\eta \psi) - (\psi^\dagger \gamma^\eta \psi)\mathscr{B}_\eta = 0. \tag{8.13}$$

In general the 0-vector component of the product of any two Clifford numbers is identical to the 0-vector component of the product of the same two Clifford numbers when the order of the product is reversed. This means that if we project out the 0-vector component of each term in Eq. (8.13), the projection of the last two terms will cancel one another out and we obtain the relation

$$((\nabla_\eta \psi^\dagger)\gamma^\eta \psi + \psi^\dagger \gamma^\eta \nabla_\eta \psi)_0 = 0. \tag{8.14}$$

For an index free Clifford number such as ψ, we know that the covariant derivative $\psi_{;\eta}$ is equal to $\nabla_\eta \psi$. Furthermore $\gamma^\alpha{}_{;\eta} = 0$. Thus Eq. (8.14) becomes

$$(\psi^\dagger{}_{,\eta}\gamma^\eta \psi + \psi^\dagger \gamma^\eta{}_{;\eta}\psi + \psi^\dagger \gamma^\eta \psi_{;\eta})_0 = 0$$

or

$$((\psi^{\dagger}\gamma^{\eta}\psi)_{;\eta})_0 = 0. \tag{8.15}$$

Since taking the covariant derivative of a p-vector is a p-vector, one may compute the covariant divergence of $\psi^{\dagger}\gamma^{\eta}\psi$ before or after projecting out the 0-vector. Therefore Eq. (8.15) can also be written as

$$((\psi^{\dagger}\gamma^{\gamma}\psi)_0)_{;\eta} = 0. \tag{8.16}$$

From Eq. (8.16), it is clear that if we define

$$J^{\eta} = e(\psi^{\dagger}\gamma^{\eta}\psi)_0 \tag{8.17}$$

then J^{η} may be interpreted as a conserved current; that is J^{η} satisfies the continuity equation:

$$J^{\eta}{}_{,\eta} = 0. \tag{8.18}$$

Suppose we again use the fact that one may exchange the order of the product of two Clifford numbers before projecting out the 0-vector component of the product without changing the result. Then

$$(\psi^{\dagger}\gamma^{\eta}\psi)_0 = (\gamma^{\eta}\psi\psi^{\dagger})_0 = \langle\gamma^{\eta}, \psi\psi^{\dagger}\rangle$$

or

$$J^{\eta} = e\langle\gamma^{\eta}, \psi\psi^{\dagger}\rangle. \tag{8.19}$$

Since $\langle\gamma^{\eta}, \gamma_{\alpha}\rangle = \delta^{\eta}_{\alpha}$, it is possible to expand the product $\psi\psi^{\dagger}$ in terms of its p-vector components and simply read off the four components of J^{η} from the 1-vector components of $\psi\psi^{\dagger}$. That is

$$e\psi\psi^{\dagger} = IA + \gamma_0 J^0 + \gamma_1 J^1 + \gamma_2 J^2 + \gamma_3 J^3 + p\text{-vectors of higher order}. \tag{8.20}$$

Using what is known as the Gordon decomposition it is also possible to interpret the 2-vector components of $\psi\psi^{\dagger}$ (Gordon 1928).

Closely following W. Gordon's computation we first note that

$$\psi^{\dagger}\gamma^{\eta}\psi = \tfrac{1}{2}(\psi^{\dagger}\gamma^{\eta}\psi) + \tfrac{1}{2}(\psi^{\dagger}\gamma^{\eta}\psi). \tag{8.21}$$

From Eq. (8.8), we have

$$\psi = \frac{i\hbar}{mc}(\gamma^{\nu}\nabla_{\nu}\psi - \gamma^{\nu}\psi\mathscr{B}_{\nu})$$

or

$$\psi = \frac{i\hbar}{mc}(\gamma^\nu \psi_{;\nu} - \gamma^\nu \psi \mathscr{B}_\nu). \tag{8.22}$$

Taking the complex reverse of this last equation gives us

$$\psi^\dagger = -\frac{i\hbar}{mc}(\psi^\dagger{}_{;\nu}\gamma^\nu + \mathscr{B}_\nu \psi^\dagger \gamma^\nu). \tag{8.23}$$

If one now substitutes the right-hand side of Eq. (8.22) into the first term on the right-hand side of Eq. (8.21) and one also substitutes the right-hand side of Eq. (8.23) into the second term on the right-hand side of Eq. (8.21), one obtains the relation

$$\begin{aligned}\psi^\dagger \gamma^\eta \psi = \frac{i\hbar}{2mc}(&\psi^\dagger \gamma^\eta \gamma^\nu \psi_{;\nu} - \psi^\dagger \gamma^\eta \gamma^\nu \psi \mathscr{B}_\nu \\ &- \psi^\dagger{}_{;\nu}\gamma^\nu \gamma^\eta \psi - \mathscr{B}_\nu \psi^\dagger \gamma^\nu \gamma^\eta \psi).\end{aligned}$$

Since $\gamma^\eta \gamma^\nu = \gamma^{\eta\nu} + g^{\eta\nu} I$, this last equation can be rewritten in the form

$$\begin{aligned}\psi^\dagger \gamma^\eta \psi = \frac{i\hbar}{2mc}[&(\psi^\dagger \gamma^{\eta\nu}\psi_{;\nu} + \psi^\dagger{}_{;\nu}\gamma^{\eta\nu}\psi) \\ &+ g^{\eta\nu}(\psi^\dagger \psi_{;\nu} - \psi^\dagger{}_{;\nu}\psi) \\ &- g^{\eta\nu}(\psi^\dagger \psi \mathscr{B}_\nu + \mathscr{B}_\nu \psi^\dagger \psi) \\ &- (\psi^\dagger \gamma^{\eta\nu}\psi \mathscr{B}_\nu - \mathscr{B}_\nu \psi^\dagger \gamma^{\eta\nu}\psi)].\end{aligned} \tag{8.24}$$

Since $\gamma^{\eta\nu}{}_{;\nu} = 0$, the first pair of terms on the right-hand side of Eq. (8.24) may be combined into a single term $(\psi^\dagger \gamma^{\eta\nu}\psi)_{;\nu}$. In addition, if we project out the 0-vector component of both sides of Eq. (8.24), the last pair of terms will cancel out. We then have

$$\begin{aligned}J^\eta &= e(\psi^\dagger \gamma^\eta \psi)_0 \\ &= \frac{ie\hbar}{2mc}((\psi^\dagger \gamma^{\eta\nu}\psi)_{;\nu})_0 \\ &\quad + \frac{ie\hbar}{2mc} g^{\eta\nu}[(\psi^\dagger \psi_{;\nu} - \psi^\dagger{}_{,\nu}\psi)_0 - (\psi^\dagger \psi \mathscr{B}_\nu + \mathscr{B}_\nu \psi^\dagger \psi)_0].\end{aligned} \tag{8.25}$$

Following the Gordon interpretation, the first term on the right-hand side of Eq. (8.25) can be interpreted as an *internal current*:

$$J^\eta_{\mathrm{INT}} = \frac{ie\hbar}{2mc}((\psi^\dagger \gamma^{\eta\nu}\psi)_{;\nu})_0 = \frac{ie\hbar}{2mc}((\gamma^{\eta\nu}\psi\psi^\dagger)_{;\nu})_0. \tag{8.26}$$

From the form of Eq. (8.26), it is clear that J^η_{INT} is covariant. The fact that J^η_{INT} also satisfies the continuity equation is not so obvious. To show that J^η_{INT} satisfies the continuity equation, it is useful to again note that the operation of computing a covariant derivative of a Clifford number commutes with the operation of projecting out the 0-vector component of the same Clifford number. Therefore

$$J^\eta_{\mathrm{INT}} = M^{\nu\eta}{}_{;\nu} \tag{8.27}$$

where

$$M^{\nu\eta} = -\frac{ie\hbar}{2mc}(\gamma^{\nu\eta}\psi\psi^\dagger)_0. \tag{8.28}$$

To show that J^η_{INT} satisfies the continuity equation, we must show that $M^{\nu\eta}{}_{;\nu;\eta} = 0$. Since $M^{\nu\eta} = -M^{\eta\nu}$, we know that $\mathcal{M} = \frac{1}{2}M^{\nu\eta}\gamma_{\nu\eta}$ is a 2-vector. From Eq. (7.17),

$$\boldsymbol{\delta}\mathcal{M} = M^{\nu\eta}{}_{;\nu}\gamma_\eta \qquad \text{and} \qquad \boldsymbol{\delta\delta}\mathcal{M} = M^{\nu\eta}{}_{;\nu;\eta}I. \tag{8.29}$$

However, from Eq. (7.20), we know that $\boldsymbol{\delta\delta}\mathcal{M} = 0$. Thus

$$J^\eta_{\mathrm{INT};\eta} = M^{\nu\eta}{}_{;\nu;\eta} = 0. \tag{8.30}$$

The second term on the right-hand side of Eq. (8.25) was interpreted by Gordon as a *convection current* J^η_{CONV}. Thus we write

$$J^\eta_{\mathrm{CONV}} = \frac{ie\hbar}{2mc}g^{\eta\nu}[(\psi^\dagger\psi_{;\nu} - \psi^\dagger{}_{;\nu}\psi)_0 - 2(\mathcal{B}_\nu\psi^\dagger\psi)_0]. \tag{8.31}$$

This also satisfies the continuity equation since

$$J^\eta = J^\eta_{\mathrm{INT}} + J^\eta_{\mathrm{CONV}} \qquad \text{and} \qquad J^\eta{}_{;\eta} = J^\eta_{\mathrm{INT};\eta} = 0. \tag{8.32}$$

The internal current J^η_{INT} is considered analogous to the current due to bound charges mentioned at the end of Section 7.2. In the same spirit the components of $M^{\nu\eta}$ are interpreted as the electric and magnetic dipole moments of the electron due to inhomogeneities in the external electromagnetic field or due to motion of the electron. For example

$$P_z = -M^{30} = \frac{ie\hbar}{2mc}(\gamma^{30}\psi\psi^\dagger)_0 \tag{8.33}$$

and

$$M_z = -M^{12} = \frac{ie\hbar}{2mc}(\gamma^{12}\psi\psi^\dagger)_0. \tag{8.34}$$

It is worthy to note that Eq. (8.34) is identical to the equation

$$\vec{M} = \frac{e}{mc}\vec{S} \tag{8.35}$$

where

$$(S_x, S_y, S_z) = \frac{i\hbar}{2}\langle\psi, (\gamma^{23}, \gamma^{31}, \gamma^{12})\psi\rangle. \tag{8.36}$$

Equation (8.35) is considered to be valid for electrons except for small radiative corrections.

With Eq. (8.20), we saw that it was possible to interpret the 1-vector component of $e\psi\psi^\dagger$. Equation (8.28) enables us to interpret the 2-vector component. According to the result of Problem 8.1, we know that

$$(\gamma^{jk}\gamma_{pq})_0 = -\delta^{jk}_{pq}.$$

Using this result along with Eq. (8.28), it is not too difficult to verify the fact that

$$e\psi\psi^\dagger = IA + J^k\gamma_k - \frac{imc}{\hbar}M^{jk}\gamma_{jk}$$
$$+ \text{ a pseudo-vector and a pseudo-scalar.} \tag{8.37}$$

From Eq. (8.37), we see that a knowledge of $\psi\psi^\dagger$ immediately gives us the total current J^k and the components of the electromagnetic dipole tensor M^{jk}. Since $J^k_{\text{INT}} = M^{jk}{}_{;j}$ and $J^k_{\text{CONV}} = J^k - J^k_{\text{INT}}$, it is clear that the internal and convection currents may also be extracted from a knowledge of the product $\psi\psi^\dagger$. This suggests to me the possibility that all measurable entities are encompassed in the product $\psi\psi^\dagger$. This in turn suggests the possibility of a c-unitary gauge transformation which is pursued in Problem 8.4.

Problem 8.1.

(1) Convince yourself that

$$\langle\hat{\gamma}^{j_1}\hat{\gamma}^{j_2}\dots\hat{\gamma}^{j_p}, \hat{\gamma}_{k_1}\hat{\gamma}_{k_2}\dots\hat{\gamma}_{k_p}\rangle = (\hat{\gamma}^{j_p}\dots\hat{\gamma}^{j_2}\hat{\gamma}^{j_1}\hat{\gamma}_{k_1}\hat{\gamma}_{k_2}\dots\hat{\gamma}_{k_p})_0$$
$$= \delta^{j_1 j_2 \dots j_p}_{k_1 k_2 \dots k_p}.$$

(2) Use the result of part (1) to show that

$$\langle\gamma^{\alpha_1\alpha_2\dots\alpha_p}, \gamma_{\beta_1\beta_2\dots\beta_p}\rangle = \delta^{\alpha_1\alpha_2\dots\alpha_p}_{\beta_1\beta_2\dots\beta_p}. \tag{8.38}$$

(Hint: you may wish to use the relation

$$\begin{aligned}\gamma^{\alpha_1\alpha_2\ldots\alpha_p} &= (W^{\alpha_1}_{j_1}W^{\alpha_2}_{j_2}\ldots W^{\alpha_p}_{j_p})\hat{\gamma}^{j_1j_2\ldots j_p}\\ &= (W^{\alpha_1}_{j_1}W^{\alpha_2}_{j_2}\ldots W^{\alpha_p}_{j_p})(\hat{\gamma}^{j_1}\hat{\gamma}^{j_2}\ldots\hat{\gamma}^{j_p}).)\end{aligned}$$

Problem 8.2. Show that if $\hat{\gamma}_0$ is represented by a Hermitian matrix and $\hat{\gamma}_1$, $\hat{\gamma}_2$, along with $\hat{\gamma}_3$ are each represented by an anti-Hermitian matrix then $\hat{\gamma}_j^\dagger = \hat{\gamma}_0\hat{\gamma}_j^*\hat{\gamma}_0$ for $j = 0, 1, 2$, or 3 and $\hat{\gamma}_j^*$ is the complex transpose of $\hat{\gamma}_j$. In addition, show that this result can be generalized to all Clifford numbers associated with Minkowski 4-space; that is $\mathscr{A}^\dagger = \hat{\gamma}_0\mathscr{A}^*\hat{\gamma}_0$.

Problem 8.3. Use the result of Problem 8.2 to show that if $\mathscr{U}$ is c-unitary and $\mathscr{U}$ is a Clifford number associated with Minkowski 4-space then $\mathscr{U}\hat{\gamma}_0\mathscr{U}^* = \hat{\gamma}_0$.

(Since in some presentations,

$$\hat{\gamma}_0 = \begin{bmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & -1 & 0\\ 0 & 0 & 0 & -1\end{bmatrix}$$

this last result implies that for Minkowski 4-space, the group of c-unitary Clifford numbers is isomorphic to the group U(2, 2).)

Problem 8.4. (A possible gauge transformation) If all physically measurable information stored in the wave function ψ is also stored in the product $\psi\psi^\dagger$, then the wave function ψ must be considered equivalent to the wave function $\psi\mathscr{U}$ where $\mathscr{U}$ is any differentiable c-unitary Clifford number. What is the consequence of this supposition?

(1) Suppose $\psi' = \psi\mathscr{U}$ or $\psi = \psi'\mathscr{U}^\dagger$. Suppose ψ is a solution of Eq. (8.8). Show ψ' is a solution of the equation

$$\gamma^\eta\nabla_\eta\psi' - \gamma^\eta\psi'\mathscr{B}'_\eta = -\frac{imc}{\hbar}\psi'$$

where

$$\mathscr{B}'_\eta = \mathscr{U}^\dagger\mathscr{B}_\eta\mathscr{U} - (\nabla_\eta\mathscr{U}^\dagger)\mathscr{U}. \tag{8.39}$$

(2) Since $\nabla_\nu I = 0$, it is clear that

$$(\nabla_\nu\mathscr{U}^\dagger)\mathscr{U} = \nabla_\nu(\mathscr{U}^\dagger\mathscr{U}) - \mathscr{U}^\dagger\nabla_\nu\mathscr{U} = -\mathscr{U}^\dagger(\nabla_\nu\mathscr{U}). \tag{8.40}$$

Use Eqs. (8.39) and (8.40) to show that

$$\begin{aligned}\nabla_\nu \mathscr{B}'_\eta - \mathscr{B}'_\eta \mathscr{B}'_\nu &= \mathscr{U}^\dagger(\nabla_\nu \mathscr{B}_\eta - \mathscr{B}_\eta \mathscr{B}_\nu)\mathscr{U} \\ &\quad + (\nabla_\nu \mathscr{U}^\dagger)\mathscr{B}_\eta \mathscr{U} + (\nabla_\eta \mathscr{U}^\dagger)\mathscr{B}_\nu \mathscr{U} - (\nabla_\nu \nabla_\eta \mathscr{U}^\dagger)\mathscr{U}.\end{aligned} \tag{8.41}$$

(3) From Eq. (8.41), it immediately follows that

$$\begin{aligned}&\nabla_\nu \mathscr{B}'_\eta - \nabla_\eta \mathscr{B}'_\nu + \mathscr{B}'_\nu \mathscr{B}'_\eta - \mathscr{B}'_\eta \mathscr{B}'_\nu \\ &\qquad = \mathscr{U}^\dagger(\nabla_\nu \mathscr{B}_\eta - \nabla_\eta \mathscr{B}_\nu + \mathscr{B}_\nu \mathscr{B}_\eta - \mathscr{B}_\eta \mathscr{B}_\nu)\mathscr{U} \\ &\qquad\quad - ((\nabla_\nu \nabla_\eta - \nabla_\eta \nabla_\nu)\mathscr{U}^\dagger)\mathscr{U}.\end{aligned}$$

Use Eq. (5.70) to show that

$$\begin{aligned}&\nabla_\nu \mathscr{B}'_\eta - \nabla_\eta \mathscr{B}'_\nu + \mathscr{B}'_\nu \mathscr{B}'_\eta - \mathscr{B}'_\eta \mathscr{B}'_\nu - \tfrac{1}{2}\mathscr{R}_{\nu\eta} \\ &\qquad = \mathscr{U}^\dagger(\nabla_\nu \mathscr{B}_\eta - \nabla_\eta \mathscr{B}_\nu + \mathscr{B}_\nu \mathscr{B}_\eta - \mathscr{B}_\eta \mathscr{B}_\nu - \tfrac{1}{2}\mathscr{R}_{\nu\eta})\mathscr{U}.\end{aligned} \tag{8.42}$$

(Comment: If $\mathscr{B}_\eta = \boldsymbol{\Gamma}_\eta - (ie/\hbar c)A_\eta I$, then

$$\nabla_\nu \mathscr{B}_\eta - \nabla_\eta \mathscr{B}_\nu + \mathscr{B}_\nu \mathscr{B}_\eta - \mathscr{B}_\eta \mathscr{B}_\nu - \tfrac{1}{2}\mathscr{R}_{\nu\eta} = -\frac{ie}{\hbar c} F_{\nu\eta} I. \tag{8.43}$$

Since $\mathscr{U}$ commutes with I, it then follows that the left-hand side of Eq. (8.42) is also equal to $-(ie/\hbar c)F_{\nu\eta}I$.

Because of the nature of the last term on the right-hand side of Eq. (8.39), it is clear that there are solutions to Eq. (8.43) other than $\boldsymbol{\Gamma}_\eta - (ie/\hbar c)A_\eta I$. It is conceivable that some of these solutions could correspond to forces other than gravitational or electromagnetic. On the other hand it may be a situation which calls for a gauge fixing condition.)

Problem 8.5. Use Eq. (8.43) and the reasoning used in Problem 5.14 to show that

$$\gamma^\eta(\nabla_\nu \mathscr{B}_\eta - \nabla_\eta \mathscr{B}_\nu + \mathscr{B}_\nu \mathscr{B}_\eta - \mathscr{B}_\eta \mathscr{B}_\nu) = \left(-\tfrac{1}{2}R_{\nu\eta} - \frac{ie}{\hbar c} F_{\nu\eta}\right)\gamma^\eta.$$

(Except for several differences in notation and sign conventions, this equation appeared in a paper by V. Fock in 1929.)

Problem 8.6. Suppose we define the *charge conjugate* $\mathscr{A}^{\mathrm{C}}$ of a Clifford number $\mathscr{A}$ by the equation

$$\mathscr{A}^{\mathrm{C}} = -J\bar{\mathscr{A}}J \tag{8.44}$$

where $\bar{\mathscr{A}}$ is the complex conjugate of $\mathscr{A}$. Suppose ψ satisfies the equation

$$\gamma^k \nabla_k \psi - \gamma^k \psi \mathscr{B}_k = -\frac{imc}{\hbar}\psi.$$

Show that

$$\gamma^k \nabla_k \psi^{\mathrm{C}} - \gamma^k \psi^{\mathrm{C}} \mathscr{B}_k^{\mathrm{C}} = -\frac{imc}{\hbar}\psi^{\mathrm{C}}.$$

(Comment: if

$$\mathscr{B}_k = \mathbf{\Gamma}_k - \frac{ie}{\hbar c} A_k I,$$

then

$$\mathscr{B}_k^{\mathrm{C}} = \mathbf{\Gamma}_k + \frac{ie}{\hbar c} A_k I.$$

Thus ψ^{C} satisfies essentially the same equation as ψ but with the charge e replaced by $-e$. If ψ represents the wave function for an electron then ψ^{C} may be interpreted as the wave function for a positron.

The charge conjugation operation defined in this problem is identical to that presented in more traditional presentations such as that which appears in *Quantum Field Theory* by Claude Itzykson and Jean-Bernard Zuber (1980). In the usual approach one takes the complex conjugate of a specific matrix representation of the Dirac matrices instead of the complex conjugate of the coefficients of the Dirac matrices. This has the effect of making the traditional definition unnecessarily complicated.)

Problem 8.7.

(1) Show $(\mathscr{A}\mathscr{B})^{\mathrm{C}} = \mathscr{A}^{\mathrm{C}}\mathscr{B}^{\mathrm{C}}$.
(2) Show $(\mathscr{A}^{\mathrm{C}})^{\mathrm{C}} = \mathscr{A}$.

Problem 8.8. In Prob. 8.6, it was noted that if $\mathscr{B}_k$ was a linear combination of an imaginary 0-vector and a real 2-vector then the application of the charge conjugate operation results in a potential with the sign of the imaginary 0-vector component reversed and the sign of the real 2-vector component unchanged. Suppose $\mathscr{B}_k$ is a generalized potential such that $\mathscr{B}_k^{\dagger} = -\mathscr{B}_k$. In that circumstance what happens when the charge conjugate operation is applied to $\mathscr{B}_k$? In particular, what happens to the sign of the imaginary 1-vector component; the real 3-vector component; and the imaginary 4-vector component of $\mathscr{B}_k$?

Problem 8.9. According to Eq. (7.58), the equation of motion for a point

charge is

$$\nabla_s \boldsymbol{u}(s) = \frac{q}{2mc^2}(\mathscr{F}(s)\boldsymbol{u}(s) - \boldsymbol{u}(s)\mathscr{F}(s)).$$

Show that applying the charge conjugate operator to this equation is equivalent to changing the sign of q or changing the sign of s.

8.2 Clifford Solutions for the Free Electron in Flat Space

For the free electron in flat space, Dirac's equation can be written as

$$\hat{\gamma}^k \frac{\hbar}{\mathrm{i}} \frac{\partial}{\partial x^k} \psi = -mc\psi. \tag{8.45}$$

Since $\hat{\gamma}^k(\partial/\partial x^k)$ maps odd p-vectors onto even p-vectors and vice versa, it is obvious that ψ must be a Clifford function with a mixture of odd and even p-vectors. Suppose one considers a possible solution of the form:

$$\psi = (I + \hat{\gamma}^k c_k) \exp\left(-\frac{i}{\hbar} p_j x^j\right), \tag{8.46}$$

where

$$(p_0, p_1, p_2, p_3) = \left(\frac{E}{c}, -p_x, -p_y, -p_z\right)$$

and

$$(x^0, x^1, x^2, x^3) = (ct, x, y, z).$$

If we now substitute the right-hand side of Eq. (8.46) into Eq. (8.45), we have

$$-(\hat{\gamma}^k p_k)(I + \hat{\gamma}^j c_j) = -mc(I + \hat{\gamma}^j c_j). \tag{8.47}$$

Equating the 1-vector components on the two sides of Eq. (8.47), one gets

$$\hat{\gamma}^k p_k = mc\hat{\gamma}^j c_j$$

and thus

$$c_j = \frac{p_j}{mc} \quad \text{for } j = 0, 1, 2, \text{and } 3.$$

Plugging this result back into Eq. (8.47) gives us

$$(\hat{\gamma}^k p_k)(\hat{\gamma}^j p_j) = m^2c^2 I$$

or alternatively

$$(\hat{\gamma}_k p^k)(\hat{\gamma}_j p^j) = m^2c^2 I$$

or

$$\frac{E^2}{c^2} - (p)^2 = m^2c^2$$

where

$$(p)^2 = \sum_{j=1}^{3} (p^j)^2 = \vec{p}\cdot\vec{p} = (p_x)^2 + (p_y)^2 + (p_z)^2.$$

This implies that

$$E = \pm\sqrt{m^2c^2 + (p)^2c^2}. \tag{8.48}$$

Letting E designate the positive square root, we get a solution in the form:

$$\psi^{(+)} = \left[I + \hat{\gamma}_k\left(\frac{p^k}{mc}\right)\right]\exp\left(-\frac{i}{\hbar}p_j x^j\right). \tag{8.49}$$

If we replace E by $-E$ and $\vec{p}$ by $-\vec{p}$, we get a second solution:

$$\psi^{(-)} = \left[I - \hat{\gamma}_k\left(\frac{p^k}{mc}\right)\right]\exp\left(\frac{i}{\hbar}p_j x^j\right). \tag{8.50}$$

Generally a wave function is considered to represent a particle with energy E if E is an eigenvalue of the operator $(-\hbar/\mathrm{i})(\partial/\partial t)$, that is if

$$-\frac{\hbar}{\mathrm{i}}\frac{\partial}{\partial t}\psi = E\psi.$$

According to this assumption, a wave function of the form that appears in Eq. (8.50) would have to be interpreted as representing a particle with negative energy. One could choose to regard such wave functions as extraneous solutions. Alternatively one could choose to regard these solutions as ammunition for the argument that Dirac's equation represents a certain amount of nonsense.

Dirac chose neither of these options. When he presented his equation in 1928 he advanced the theory that these negative energy solutions implied the existence of antiparticles. Such prominent men as Niels Bohr and Wolfgang Pauli ridiculed him for such wild speculation. However, Dirac had to put up with this criticism only for a few years. In 1932, Carl D. Anderson discovered the positron and Dirac was awarded the Nobel Prize the following year.

It is useful to observe that

$$\hat{\gamma}_k\left(\frac{p^k}{mc}\right) = \mathscr{B}\hat{\gamma}_0\mathscr{B}^{-1}, \tag{8.51}$$

where $\mathscr{B}$ is a boost operator. In particular

$$\mathscr{B} = \exp\left(\frac{\phi}{2} J\hat{u}\right), \tag{8.52}$$

$$\hat{u} = \hat{\gamma}_{23}u^1 + \hat{\gamma}_{31}u^2 + \hat{\gamma}_{12}u^3,$$

$$J\hat{u} = \gamma_{10}u^1 + \gamma_{20}u^2 + \gamma_{30}u^3,$$

$$\cosh\phi = u^0 = \frac{1}{\sqrt{1-(v/c)^2}}$$

$$\sinh\phi = u = \frac{v/c}{\sqrt{1-(v/c)^2}}.$$

Using Eqs. (8.49)–(8.51), we get

$$\psi^{(+)} = \mathscr{B}(I + \hat{\gamma}_0)\mathscr{B}^{-1}\exp\left(-\frac{\mathrm{i}}{\hbar}p_k x^k\right) \tag{8.53}$$

and

$$\psi^{(-)} = \mathscr{B}(I - \hat{\gamma}_0)\mathscr{B}^{-1}\exp\left(\frac{\mathrm{i}}{\hbar}p_k x^k\right). \tag{8.54}$$

Because of the form of Dirac's equation in flat space, it is possible to multiply any solution on the right-hand side by a constant Clifford number to obtain another solution. In this fashion, the most general solutions can be written in the form

$$\psi^{(+)} = \exp\left(-\frac{\mathrm{i}}{\hbar}p_k x^k\right)\mathscr{B}(I + \hat{\gamma}_0)\mathscr{C} \tag{8.55}$$

and

$$\psi^{(-)} = \exp\left(\frac{\mathrm{i}}{\hbar}p_k x^k\right)\mathscr{B}(I - \hat{\gamma}_0)\mathscr{C}, \tag{8.56}$$

where $\mathscr{C}$ is an arbitrary constant Clifford number.

If we now insist that ψ be an eigenfunction of the spin operator $(\mathrm{i}\hbar/2)\hat{\gamma}_{12}$

in the rest frame and $\mathscr{B}(\mathrm{i}\hbar/2)\hat{\gamma}_{12}\mathscr{B}^{-1}$ in the laboratory frame, it should be noted that

$$\frac{\mathrm{i}\hbar}{2}\hat{\gamma}_{12}(I \pm \mathrm{i}\hat{\gamma}_{12}) = \pm\frac{\hbar}{2}(I \pm \mathrm{i}\hat{\gamma}_{12}).$$

Incorporating the spin terms into the solutions gives us

$$\psi^{(+\uparrow)} = \exp\left(-\frac{\mathrm{i}}{\hbar}p_k x^k\right)\mathscr{B}(I + \hat{\gamma}_0)(I + \mathrm{i}\hat{\gamma}_{12})\mathscr{C}; \tag{8.57}$$

$$\psi^{(+\downarrow)} = \exp\left(-\frac{\mathrm{i}}{\hbar}p_k x^k\right)\mathscr{B}(I + \hat{\gamma}_0)(I - \mathrm{i}\hat{\gamma}_{12})\mathscr{C}; \tag{8.58}$$

$$\psi^{(-\uparrow)} = \exp\left(\frac{\mathrm{i}}{\hbar}p_k x^k\right)\mathscr{B}(I - \hat{\gamma}_0)(I + \mathrm{i}\hat{\gamma}_{12})\mathscr{C}; \tag{8.59}$$

$$\psi^{(-\downarrow)} = \exp\left(\frac{\mathrm{i}}{\hbar}p_k x^k\right)\mathscr{B}(I - \hat{\gamma}_0)(I - \mathrm{i}\hat{\gamma}_{12})\mathscr{C}. \tag{8.60}$$

One advantage to this formulation is that it is easy to represent a state with arbitrary spin direction. If (s^1, s^2, s^3) represents a vector of unit length then we can represent the state of a free electron with spin in that direction as follows:

$$\psi^{(+)} = \exp\left(\frac{\mathrm{i}}{\hbar}p_k x^k\right)\mathscr{B}(I + \hat{\gamma}_0)(I + \mathrm{i}\hat{s})\mathscr{C} \tag{8.61}$$

where

$$\hat{s} = \hat{\gamma}_{23}s^1 + \hat{\gamma}_{31}s^2 + \hat{\gamma}_{12}s^3. \tag{8.62}$$

It should be understood that $\hat{s}$ represents the spin direction in the rest frame of the particle. (See Problem 8.14.)

It is useful to note that $\frac{1}{2}(I \pm \hat{\gamma}_0)$ and $\frac{1}{2}(I + \mathrm{i}\hat{s})$ are commuting projection operators; that is

$$\tfrac{1}{2}(I \pm \hat{\gamma}_0)\tfrac{1}{2}(I \pm \hat{\gamma}_0) = \tfrac{1}{2}(I \pm \hat{\gamma}_0); \tag{8.63}$$

$$\tfrac{1}{2}(I + \hat{\gamma}_0)\tfrac{1}{2}(I - \hat{\gamma}_0) = \tfrac{1}{2}(I - \hat{\gamma}_0)\tfrac{1}{2}(I + \hat{\gamma}_0) = 0; \tag{8.64}$$

$$\tfrac{1}{2}(I + \mathrm{i}\hat{s})\tfrac{1}{2}(I + \mathrm{i}\hat{s}) = \tfrac{1}{2}(I + \mathrm{i}\hat{s}); \tag{8.65}$$

and if $P = \frac{1}{2}(I \pm \hat{\gamma}_0)\frac{1}{2}(I + \mathrm{i}\hat{s})$ then $(P)^2 = P$.

To compute the 4-current, we note that from Eq. (8.61), we have, except

for an appropriate normalizing constant:

$$e\psi^{(+)}\psi^{(+)\dagger} = e\mathscr{B}(I + \hat{\gamma}_0)(I + i\hat{s})\mathscr{C}\mathscr{C}^\dagger(I + i\hat{s})(I + \hat{\gamma}_0)\mathscr{B}^{-1}. \tag{8.66}$$

It will be shown in the next section that there is no loss of generality if $\mathscr{C}$ is required to be a linear combination of real even order p-vectors. From Eq. (A1.8), we know that for such a Clifford number, $\mathscr{C}\mathscr{C}^\dagger = aI + bJ$ where a and b are real numbers. Also J anticommutes with $\hat{\gamma}_0$ so

$$(I + \hat{\gamma}_0)J(I + \hat{\gamma}_0) = J(I - \hat{\gamma}_0)(I + \hat{\gamma}_0) = 0.$$

Using this result along with Eqs. (8.63) and (8.65), Eq. (8.66) becomes

$$e\psi^{(+)}\psi^{(+)\dagger} = 4ae\mathscr{B}(I + \hat{\gamma}_0)(I + i\hat{s})\mathscr{B}^{-1}. \tag{8.67}$$

The 1-vector component of this is

$$\mathscr{J}^{(+)} = J^k\hat{\gamma}_k = 4ae\mathscr{B}\hat{\gamma}_0\mathscr{B}^{-1} = 4ae(\hat{\gamma}_k u^k), \tag{8.68}$$

where the u^k's are the components of the world velocity.

For positrons, a similar calculation gives

$$e\psi^{(-)}\psi^{(-)\dagger} = 4ae\mathscr{B}(I - \hat{\gamma}_0)(I + i\hat{s})\mathscr{B}^{-1} \tag{8.69}$$

and

$$\mathscr{J}^{(-)} = J^k\hat{\gamma}_k = -4ae\mathscr{B}\hat{\gamma}_0\mathscr{B}^{-1} = -4ae\hat{\gamma}_k u^k. \tag{8.70}$$

Problem 8.10. Show that for the free electron, the phase velocity $v_{\text{PHASE}} = E/p$. Show that the velocity of the 4-current or the velocity associated with the boost operator is pc^2/E. (Note: $v_{\text{PHASE}}v_{\text{CURRENT}} = c^2$, $v_{\text{CURRENT}} < c$, and $v_{\text{PHASE}} > c$.)

Problem 8.11. Show that

$$\exp(-iw)(I \pm \hat{\gamma}_{12}) = \exp(\pm\hat{\gamma}_{12}w)(I \pm \hat{\gamma}_{12}).$$

(This means that the term "$\exp[-(i/\hbar)p_k x^k]$" that appears in the free electron solution may be interpreted as a time-dependent rotation operator about the spin axis. I have no idea whether or not this has any significance.)

Problem 8.12. Use Eqs. (8.67) and (8.37) to determine both the electric and magnetic dipoles of the free electron. Observe that the electric dipole is induced by the motion and is perpendicular to both the direction of the magnetic dipole and the direction of motion.

Problem 8.13. Apply the charge conjugate operation defined in Problem 8.6 to the free electron solutions of Eqs. (8.57)–(8.60) and thereby demonstrate that

$$(\psi^{(+\uparrow)})^{C} = \psi^{(-\downarrow)} \qquad \text{and} \qquad (\psi^{(+\downarrow)})^{C} = \psi^{(-\uparrow)}.$$

Problem 8.14. Suppose the $\psi^{(+)}$ is the solution for the free electron written in the form of Eq. (8.61). Show that $\mathscr{B}(i\hbar/2)\hat{s}\mathscr{B}^{-1}\psi^{(+)} = (\hbar/2)\psi^{(+)}$.

8.3 A Canonical Form for Solutions to Dirac's Equation in Flat Space

The free particle solutions to Dirac's equation suggest that one can find solutions of the form:

$$\psi = \mathbf{\Phi}(I \pm i\hat{\gamma}_{12})(I \pm \gamma_0)\mathscr{C}. \tag{8.71}$$

This is indeed the case. In itself this does not say much. If $\mathbf{\Phi}$ is a solution, then we can multiply it on the right-hand side by any constant Clifford number and the resulting product must also be a solution of Dirac's equation. Since $(I \pm i\hat{\gamma}_{12})(I \pm \hat{\gamma}_0)\mathscr{C}$ is such a constant Clifford number, it is clear that the product $\Phi(I \pm i\hat{\gamma}_{12})(I \pm \hat{\gamma}_0)\mathscr{C}$ is a solution if $\mathbf{\Phi}$ is a solution. Furthermore an arbitrary solution can be written as a linear combination of such forms since

$$\begin{aligned}\Phi\mathscr{C} = {}& \tfrac{1}{4}\mathbf{\Phi}(I + i\hat{\gamma}_{12})(I + \hat{\gamma}_0)\mathscr{C} + \tfrac{1}{4}\mathbf{\Phi}(I - i\hat{\gamma}_{12})(I + \hat{\gamma}_0)\mathscr{C} \\ & + \tfrac{1}{4}\mathbf{\Phi}(I + i\hat{\gamma}_{12})(I - \hat{\gamma}_0)\mathscr{C} + \tfrac{1}{4}\mathbf{\Phi}(I - i\hat{\gamma}_{12})(I - \hat{\gamma}_0)\mathscr{C}.\end{aligned}$$

What makes the solutions of the forms that appear in Eq. (8.71) useful is the fact that the Clifford number $\mathbf{\Phi}$ can be cast in a form that lends itself to physical interpretation. First of all, $\mathbf{\Phi}$ can be written as a linear combination of real p-vectors of even order. To prove this, we will consider the case for which

$$\psi = \mathbf{\Phi}(I + i\hat{\gamma}_{12})(I + \hat{\gamma}_0)\mathscr{C}.$$

(It should become clear to the reader that the argument used for this case can be generalized to all four cases that appear in Eq. (8.71).)

Any complex Clifford number can be written as a linear combination of a real Clifford number and an imaginary Clifford number. Thus $\mathbf{\Phi} = \mathbf{\Phi}_1 + i\mathbf{\Phi}_2$ where both $\mathbf{\Phi}_1$ and $\mathbf{\Phi}_2$ are real. However, $(I + i\hat{\gamma}_{12}) = i\hat{\gamma}_{12}(I + i\hat{\gamma}_{12})$. Therefore

$$\begin{aligned}\mathbf{\Phi}(I + i\hat{\gamma}_{12})(I + \hat{\gamma}_0)\mathscr{C} &= (\mathbf{\Phi}_1 + i\mathbf{\Phi}_2)(I + i\hat{\gamma}_{12})(I + \hat{\gamma}_0)\mathscr{C} \\ &= (\mathbf{\Phi}_1 - \mathbf{\Phi}_2\hat{\gamma}_{12})(I + i\hat{\gamma}_{12})(I + \hat{\gamma}_0)\mathscr{C}.\end{aligned} \tag{8.72}$$

Thus if $\mathbf{\Phi}$ is complex, it can be replaced by the real Clifford number $\mathbf{\Phi}_1 - \mathbf{\Phi}_2\hat{\gamma}_{12}$.

As stated above it is also possible to require that $\mathbf{\Phi}$ be a linear combination of even order p-vectors. To show this, we note that an arbitrary Clifford number can be written as a sum of two Clifford numbers. One consists of a linear combination of even order p-vectors and the other consists of a linear combination of odd order p-vectors. That is

$$\mathbf{\Phi} = \mathbf{\Phi}_{\mathrm{EVEN}} + \mathbf{\Phi}_{\mathrm{ODD}}.$$

On the other hand $(I + \hat{\gamma}_0) = \hat{\gamma}_0(I + \hat{\gamma}_0)$. Thus

$$\begin{aligned}\mathbf{\Phi}(I + \mathrm{i}\hat{\gamma}_{12})(I + \hat{\gamma}_0)\mathscr{C} &= (\mathbf{\Phi}_{\mathrm{EVEN}} + \mathbf{\Phi}_{\mathrm{ODD}})(I + \mathrm{i}\hat{\gamma}_{12})(I + \hat{\gamma}_0)\mathscr{C} \\ &= (\mathbf{\Phi}_{\mathrm{EVEN}} + \mathbf{\Phi}_{\mathrm{ODD}}\hat{\gamma}_0)(I + \mathrm{i}\hat{\gamma}_{12})(I + \hat{\gamma}_0)\mathscr{C}.\end{aligned}$$

Therefore if $\mathbf{\Phi}$ has some odd order components, $\mathbf{\Phi}$ can be replaced by $\mathbf{\Phi}_{\mathrm{EVEN}} + \mathbf{\Phi}_{\mathrm{ODD}}\hat{\gamma}_0$ which consists of even order p-vectors only.

Clearly the same comments made above on the Clifford function $\mathbf{\Phi}$ also apply to the Clifford number $\mathscr{C}$. That is $\mathscr{C}$ may be restricted to be a linear combination of real even order p-vectors without losing any generality. In the last section, it was pointed out that with this restriction, the choice of $\mathscr{C}$ has no impact on the product $\psi\psi^\dagger$ except for a possible normalizing constant. On the other hand, the arbitrariness of $\mathscr{C}$ is critical in scattering theory where one must compute products of the type $\psi_1\psi_2^\dagger$ where ψ_1 and ψ_2 represent Clifford wave functions for different particles.

The fact that $\mathbf{\Phi}$ may be restricted to a linear combination of real even order p-vectors enables us to extract a physical interpretation from it. From Theorem A1.1 and Theorem A1.2 of Section A.1 of the Appendix, we know that if $\mathbf{\Phi\Phi}^\dagger \neq 0$, then

$$\mathbf{\Phi} = N\mathscr{B}\mathscr{R}\mathscr{J}, \tag{8.73}$$

where N is a normalizing constant, $\mathscr{B}$ is a boost operator, $\mathscr{R}$ is a rotation operator, and $\mathscr{J}$ is a duality rotation which has the form

$$\mathscr{J} = \exp\left(\frac{\omega}{2}J\right). \tag{8.74}$$

One may ask what is the significance of the duality rotation operator. When one computes $\psi\psi^\dagger$, one has

$$\psi\psi^\dagger = \frac{N^2}{4}\mathscr{B}\mathscr{R}\mathscr{J}(I \pm \hat{\gamma}_0)(I + \mathrm{i}\hat{s})\mathscr{J}^\dagger\mathscr{R}^\dagger\mathscr{B}^\dagger. \tag{8.75}$$

Since J commutes with $\hat{s}$, the duality rotation operator has a direct effect

only on $(I \pm \hat{\gamma}_0)$. We note that

$$\mathscr{I}\hat{\gamma}_0\mathscr{I}^\dagger = \exp\left(\frac{\omega}{2}J\right)\hat{\gamma}_0\exp\left(\frac{\omega}{2}J\right)$$
$$= \exp\left(\frac{\omega}{2}J\right)\exp\left(-\frac{\omega}{2}J\right)\hat{\gamma}_0 = \hat{\gamma}_0.$$

Therefore

$$\mathscr{I}(I \pm \hat{\gamma}_0)\mathscr{I}^\dagger = \exp(\omega J) \pm \hat{\gamma}_0$$
$$= (I\cos\omega + J\sin\omega \pm \hat{\gamma}_0). \tag{8.76}$$

Plugging this result back into Eq. (8.75), we have

$$\psi\psi^\dagger = \frac{N^2}{4}\mathscr{B}\mathscr{R}(I\cos\omega \pm \hat{\gamma}_0 + \mathrm{i}\hat{s}\cos\omega + \mathrm{i}J\hat{s}\sin\omega \pm \mathrm{i}\hat{\gamma}_0\hat{s} + J\sin\omega)\mathscr{R}^{-1}\mathscr{B}^{-1}. \tag{8.77}$$

From Eq. (8.37), the electromagnetic dipole 2-vector is

$$\mathscr{M} = \tfrac{1}{2}M^{jk}\hat{\gamma}_{jk} = -\frac{eN^2\hbar}{8mc}\mathscr{B}\mathscr{R}(\hat{s}\cos\omega + J\hat{s}\sin\omega)\mathscr{R}^{-1}\mathscr{B}^{-1}. \tag{8.78}$$

This suggests that

$$-\frac{eN^2\hbar}{8mc}(\hat{s}\cos\omega + J\hat{s}\sin\omega)$$

may be interpreted as the electromagnetic dipole 2-vector in the rest frame of the electron. In particular if $\hat{s} = \hat{\gamma}_{12}$, then in the frame of the electron,

$$M_z = -M^{12} = \frac{eN^2\hbar}{8mc}\cos\omega \tag{8.79}$$

and

$$P_z = -M^{30} = \frac{eN^2\hbar}{8mc}\sin\omega. \tag{8.80}$$

For the free electron, $\omega = 0$. However, for the electron associated with a hydrogen atom, this is not the case. Presumably the electric dipole is induced by the inhomogeneous electric field due to the nucleus. It should be pointed out that both N^2 and ω have a spatial dependence. Thus M^{jk} has a spatial dependence.

Problem 8.15. Suppose M^{jk} is the electromagnetic dipole tensor, that is

$$[M^{jk}] = \begin{bmatrix} 0 & P_x & P_y & P_z \\ -P_x & 0 & -M_z & M_y \\ -P_y & M_z & 0 & -M_x \\ -P_z & -M_y & M_x & 0 \end{bmatrix}.$$

Suppose further that $P_x = P_y = P_z = 0$. Apply an arbitrary boost operator to the 2-vector $\mathscr{M} = \frac{1}{2}M^{jk}\gamma_{jk}$ and show that in the "boosted" frame the electric dipole is usually not zero but it is always perpendicular to the magnetic dipole.

Problem 8.16. In the usual approach to Dirac's equation, one uses a specific representation for the Dirac matrices and then seeks out vector solutions. These vector solutions are then used to compute expectation values. This problem is intended to provide the reader with a comparison of this approach and the approach presented in this text.

In virtually all representations used in actual computations $\hat{\gamma}^0$ is Hermitian and $\hat{\gamma}^j$ is anti-Hermitian for $j = 1, 2,$ and 3.

(1) For such a representation show that if ψ is an arbitrary Clifford number then

$$\psi^*\hat{\gamma}^0 = \hat{\gamma}^0\psi^\dagger \tag{8.81}$$

where ψ^* is the Hermitian conjugate of ψ.

From Eq. (8.81), it follows that

$$\hat{\gamma}^0\psi^\dagger\mathscr{A}\psi = \psi^*\hat{\gamma}^0\mathscr{A}\psi. \tag{8.82}$$

If $\psi^{(+)} = \Phi(I \pm i\hat{\gamma}_{12})(I + \hat{\gamma}_0)$, then $\hat{\gamma}^0\psi^{(+)\dagger} = \psi^{(+)\dagger}$ and

$$\psi^{(+)*}\hat{\gamma}^0\mathscr{A}\psi^{(+)} = \psi^{(+)\dagger}\mathscr{A}\psi^{(+)}. \tag{8.83}$$

If $\psi^{(-)}$ is a negative energy solution—that is if

$$\psi^{(-)} = \Phi(\pm i\hat{\gamma}_{12})(I - \hat{\gamma}_0), \text{ then } \hat{\gamma}^0\psi^{(-)\dagger} = -\psi^{(-)\dagger}$$

and

$$\psi^{(-)*}\hat{\gamma}^0\mathscr{A}\psi^{(-)} = -\psi^{(-)\dagger}\mathscr{A}\psi^{(-)}. \tag{8.84}$$

When dealing with a specific representation of Dirac matrices, any column of a solution matrix ψ may be considered to be a column vector solution. In the usual formulation, one computes expectations in terms of these column vector solutions. For example, one obtains the components of a "probability current" J^k from the defini-

tion: $J^k = (\vec{\psi}^* \hat{\gamma}^0 \hat{\gamma}^k \vec{\psi})$ where $\vec{\psi}$ is a column vector solution. More generally, the expectation of a matrix operator $\mathscr{A}$ is computed by computing $\langle \mathscr{A} \rangle = (\vec{\psi}^* \hat{\gamma}^0 \mathscr{A} \vec{\psi})$.

(2) Using the representation of Eq. (2.8), compute each of the four matrices:

$$\tfrac{1}{4}(I \pm i\hat{\gamma}_{12})(I \pm \hat{\gamma}_0).$$

(3) Use the result of part (2) to show that if ψ is in one of the four canonical forms $\mathbf{\Phi}(I \pm i\hat{\gamma}_{12})(I \pm \hat{\gamma}_0)$, then for the matrix representation of Eq. (2.8), ψ has only one nonzero column.

(4) Show that if ψ is in one of the four canonical forms and $\vec{\psi}$ is its nonzero column then

$$\begin{aligned}(\vec{\psi}^* \hat{\gamma}^0 \mathscr{A} \vec{\psi}) &= 4(\psi^* \hat{\gamma}^0 \mathscr{A} \psi)_0 = 4(\hat{\gamma}^0 \psi^\dagger \mathscr{A} \psi)_0 \\ &= \pm 4(\psi^\dagger \mathscr{A} \psi)_0 = \pm 4(\mathscr{A} \psi \psi^\dagger)_0 = \pm 4\langle \psi, \mathscr{A} \psi \rangle.\end{aligned}$$

The + sign occurs for the positive energy solutions and the − sign occurs for the negative energy solutions. Thus we see that the two formalisms give essentially the same expectation values. The factor of 4 disappears when one assigns different normalization conventions to the two formalisms. In the standard formalism, $(\vec{\psi}^* \hat{\gamma}^0 \hat{\gamma}^k \vec{\psi})$ is interpreted as a probability current and the sign is adjusted "by hand" when the probability current is multiplied by $\pm e$ to give the charge current. In the formalism of this text, the computed current is interpreted as a charge current and the sign difference between particles and antiparticles arises from the formalism automatically.

8.4 Spherical Harmonic Clifford Functions

Before solving Dirac's equation for the hydrogen atom in the next section, we will examine the Clifford analogue of the spherical harmonic functions for dealing with Dirac's equation when the potential is spherically symmetric.

If it is understood that

$$-\frac{\hbar}{i}\frac{\partial}{\partial t}\psi = E\psi,$$

then the time-independent version of Dirac's equation for an electron in a Coulomb potential is

$$\hat{\gamma}^0\left(-\frac{E}{c} - \frac{Ze^2}{4\pi c r}\right)\psi + \sum_{k=1}^{3} \hat{\gamma}^k\left(\frac{\hbar}{i}\frac{\partial}{\partial x^k}\right)\psi = -mc\psi. \tag{8.85}$$

This equation can be rewritten as an eigenvalue equation by multiplying

both sides by $c\hat{\gamma}^0$ and regrouping the terms. The result is

$$\boldsymbol{H}\psi = E\psi, \tag{8.86}$$

where

$$\boldsymbol{H} = \sum_{k=1}^{3} \hat{\gamma}^{0k}\left(\frac{\hbar c}{\mathrm{i}}\frac{\partial}{\partial x^k}\right) - \frac{Ze^2}{4\pi r} + \hat{\gamma}^0 mc^2. \tag{8.87}$$

It can be shown that

$$\boldsymbol{J}_k\boldsymbol{H} = \boldsymbol{H}\boldsymbol{J}_k \quad \text{for } k = 1, 2, \text{and } 3, \tag{8.88}$$

where

$$\boldsymbol{J}_k = \boldsymbol{L}_k + \boldsymbol{S}_k, \tag{8.89}$$

$$\boldsymbol{L}_1 = \frac{\hbar}{\mathrm{i}}\left(x^2\frac{\partial}{\partial x^3} - x^3\frac{\partial}{\partial x^2}\right), \tag{8.90}$$

$$\boldsymbol{L}_2 = \frac{\hbar}{\mathrm{i}}\left(x^3\frac{\partial}{\partial x^1} - x^1\frac{\partial}{\partial x^3}\right), \tag{8.91}$$

$$\boldsymbol{L}_3 = \frac{\hbar}{\mathrm{i}}\left(x^1\frac{\partial}{\partial x^2} - x^2\frac{\partial}{\partial x^1}\right), \tag{8.92}$$

$$\boldsymbol{S}_1 = \frac{\mathrm{i}\hbar}{2}\hat{\gamma}_{23}, \tag{8.93}$$

$$\boldsymbol{S}_2 = \frac{\mathrm{i}\hbar}{2}\hat{\gamma}_{31}, \tag{8.94}$$

$$\boldsymbol{S}_3 = \frac{\mathrm{i}\hbar}{2}\hat{\gamma}_{12}. \tag{8.95}$$

From Eq. (8.88), it follows that if $\boldsymbol{H}\psi = E\psi$, then

$$\boldsymbol{H}(\boldsymbol{J}_3\psi) = \boldsymbol{J}_3\boldsymbol{H}\psi = \boldsymbol{J}_3E\psi = E(\boldsymbol{J}_3\psi).$$

This means that if ψ is a solution of the time-independent Dirac equation with energy eigenvalue E, then $\boldsymbol{J}_3\psi$ is also a solution with the same eigenvalue. This in turn implies that any solution to the time-independent Dirac equation can be decomposed into a linear combination of Clifford functions each of which is an eigenfunction of $\boldsymbol{J}_3$.

Since neither $\boldsymbol{J}_1$ nor $\boldsymbol{J}_2$ commutes with $\boldsymbol{J}_3$, it is not possible to find Clifford functions which are simultaneously eigenfunctions of $\boldsymbol{H}$, $\boldsymbol{J}_3$, $\boldsymbol{J}_1$, and $\boldsymbol{J}_2$.

However, J_3 does commute with $|J|^2$ where

$$|J|^2 = (J_1)^2 + (J_2)^2 + (J_3)^2. \tag{8.96}$$

Thus if H corresponds to a Coulomb potential or any spherically symmetric potential, it is possible to find Clifford functions which are simultaneously eigenfunctions of H, J_3, and $|J|^2$. For this reason, it is useful to determine those Clifford functions which are simultaneously eigenfunctions of J_3 and $|J|^2$. To pursue this goal, one should note that the eigenfunctions of L_3 and $|L|^2$ are well known as *spherical harmonics* (Jackson 1962, pp. 54–69; or Abramowitz and Stegun 1965, pp. 332–341). These functions are defined by

$$Y_{l,\lambda}(\theta, \phi) = \sqrt{\frac{2l+1}{4\pi}\frac{(l-\lambda)!}{(l+\lambda)!}}\, P_l^{\lambda}(\cos\theta)e^{i\lambda\phi}, \tag{8.97}$$

where P_l^{λ} is an *associated Legendre function* which can be defined by the equation:

$$P_l^{\lambda}(u) = \frac{(-1)^{\lambda}}{2^l l!}(1-u^2)^{\lambda/2}\frac{d^{l+\lambda}}{du^{l+\lambda}}(u^2-1)^l. \tag{8.98}$$

(Here l is a nonnegative integer and λ is any integer between $-l$ and $+l$.)

These functions satisfy the relation

$$L_3 Y_{l,\lambda} = \hbar\lambda Y_{l,\lambda} \tag{8.99}$$

and also

$$|L|^2 Y_{l,\lambda} = \hbar^2 l(l+1) Y_{l,\lambda}. \tag{8.100}$$

We note that

$$|J|^2 = |L|^2 + i\hbar(\hat{\gamma}_{23}L_1 + \hat{\gamma}_{31}L_2 + \hat{\gamma}_{12}L_3) + \frac{3\hbar^2}{4}I. \tag{8.101}$$

Each of the operators L_1, L_2, and L_3 map linear combinations of spherical harmonics for some fixed value of l into linear combinations of spherical harmonics of the same type. Because of this, Clifford eigenfunctions of $|J|^2$ and J_3 may be written in the form: $\mathcal{Y} = Y_{l,\lambda}\mathcal{A}^{\lambda}$, where the $\mathcal{A}^{\lambda}$'s are constant Clifford numbers—at least not functions of θ or ϕ.

Since $\mathcal{A}^{\lambda} = \frac{1}{2}(I + i\hat{\gamma}_{12})\mathcal{A}^{\lambda} + \frac{1}{2}(I - i\hat{\gamma}_{12})\mathcal{A}^{\lambda}$, we see that

$$J_3\mathcal{Y} = \left(L_3 + \frac{i\hbar}{2}\hat{\gamma}_{12}\right)Y_{l,\lambda}\tfrac{1}{2}(I + i\hat{\gamma}_{12})\mathcal{A}^{\lambda} + \left(L_3 + \frac{i\hbar}{2}\hat{\gamma}_{12}\right)Y_{l,\lambda}\tfrac{1}{2}(I - i\hat{\gamma}_{12})\mathcal{A}^{\lambda}.$$

Since $i\hat{\gamma}_{12}(I \pm i\hat{\gamma}_{12}) = \pm(I \pm i\hat{\gamma}_{12})$, this last equation becomes

$$J_3\mathscr{Y} = \hbar(\lambda + \tfrac{1}{2})Y_{l,\lambda}\tfrac{1}{2}(I + i\hat{\gamma}_{12})\mathscr{A}^{\lambda} + \hbar(\lambda - \tfrac{1}{2})Y_{l,\lambda}\tfrac{1}{2}(I - i\hat{\gamma}_{12})\mathscr{A}^{\lambda}.$$

If $J_3\mathscr{Y} = m\hbar\mathscr{Y}$, then

$$\mathscr{Y} = Y_{l,m-\frac{1}{2}}\tfrac{1}{2}(I + i\hat{\gamma}_{12})\mathscr{A} + Y_{l,m+\frac{1}{2}}\tfrac{1}{2}(I - i\hat{\gamma}_{12})\mathscr{A}'.$$

It will be useful to replace $\mathscr{A}'$ by $\hat{\gamma}_{31}\mathscr{B}$. If we pause to note that $(I - i\hat{\gamma}_{12})\hat{\gamma}_{31} = \hat{\gamma}_{31}(I + i\hat{\gamma}_{12})$, our equation for $\mathscr{Y}$ becomes

$$\mathscr{Y} = Y_{l,m-\frac{1}{2}}\tfrac{1}{2}(I + i\hat{\gamma}_{12})\mathscr{A} + Y_{l,m+\frac{1}{2}}\hat{\gamma}_{31}\tfrac{1}{2}(I + i\hat{\gamma}_{12})\boldsymbol{B}. \tag{8.102}$$

Now we need to adjust the Clifford numbers $\mathscr{A}$ and $\mathscr{B}$ so that $\mathscr{Y}$ is also an eigenfunction of $|\boldsymbol{J}|^2$. We note that without this adjustment $\mathscr{Y}$ is already an eigenfunction of $|\boldsymbol{L}|^2$. From Eq. (8.101), we see that for $\mathscr{Y}$ to be an eigenfunction of $|\boldsymbol{J}|^2$, it must also be an eigenfunction of

$$\hat{\gamma}_0\hbar\boldsymbol{k} = i\hat{\gamma}_{23}\boldsymbol{L}_1 + i\hat{\gamma}_{31}\boldsymbol{L}_2 + i\hat{\gamma}_{12}\boldsymbol{L}_3 + \hbar I. \tag{8.103}$$

This may be rewritten in the form:

$$\hat{\gamma}_0\hbar\boldsymbol{k} = \hat{\gamma}_{31}\tfrac{1}{2}(I + i\hat{\gamma}_{12})\boldsymbol{L}_+ - \hat{\gamma}_{31}\tfrac{1}{2}(I - i\hat{\gamma}_{12})\boldsymbol{L}_- + i\hat{\gamma}_{12}\boldsymbol{L}_3 + \hbar I, \tag{8.104}$$

where

$$\boldsymbol{L}_+ = \boldsymbol{L}_1 + i\boldsymbol{L}_2, \tag{8.105}$$

and

$$\boldsymbol{L}_- = \boldsymbol{L}_1 - i\boldsymbol{L}_2. \tag{8.106}$$

(Labeling our operator by $\hat{\gamma}_0\hbar\boldsymbol{k}$ may seem strange, but the operator $\boldsymbol{k}$ will prove to be helpful in its own right.)

The operators $\boldsymbol{L}_+$ and $\boldsymbol{L}_-$ are known respectively as *step up* and *step down* operators. It can be shown with some difficulty that

$$\boldsymbol{L}_+ Y_{l,\lambda} = \hbar\sqrt{(l - \lambda)(l + \lambda + 1)}\, Y_{l,\lambda+1} \tag{8.107}$$

and

$$\boldsymbol{L}_- Y_{l,\lambda} = \hbar\sqrt{(l + \lambda)(l - \lambda + 1)}\, Y_{l,\lambda-1}. \tag{8.108}$$

From Eqs. (8.102), (8.103), (8.107), and (8.108), we have

$$\begin{aligned}\hat{\gamma}_0\hbar k\mathscr{Y} &= \hbar\sqrt{(l+\tfrac{1}{2})^2-m^2}\,Y_{l,m-\frac{1}{2}}\tfrac{1}{2}(I+\mathrm{i}\hat{\gamma}_{12})\mathscr{B} \\ &\quad+\hbar(m-\tfrac{1}{2})Y_{l,m-\frac{1}{2}}\tfrac{1}{2}(I+\mathrm{i}\hat{\gamma}_{12})\mathscr{A}. \\ &\quad+\hbar\sqrt{(l+\tfrac{1}{2})^2-m^2}\,Y_{l,m+\frac{1}{2}}\hat{\gamma}_{31}\tfrac{1}{2}(I+\mathrm{i}\hat{\gamma}_{12})\mathscr{A} \\ &\quad-\hbar(m+\tfrac{1}{2})\,Y_{l,m+\frac{1}{2}}\hat{\gamma}_{31}\tfrac{1}{2}(I+\mathrm{i}\hat{\gamma}_{12})\mathscr{B}+\hbar\mathscr{Y} \\ &= \hbar k\mathscr{Y}.\end{aligned}$$

Equating coefficients of $Y_{l,m-\frac{1}{2}}\frac{1}{2}(I+\mathrm{i}\hat{\gamma}_{12})$ and then coefficients of

$$Y_{l,m+\frac{1}{2}}\hat{\gamma}_{31}\tfrac{1}{2}(I+\mathrm{i}\hat{\gamma}_{12}),$$

we have

$$\sqrt{(l+\tfrac{1}{2})^2-m^2}\,\mathscr{B}+(m-\tfrac{1}{2})\mathscr{A}+\mathscr{A}=k\mathscr{A}$$

and

$$\sqrt{(l+\tfrac{1}{2})^2-m^2}\,\mathscr{A}-(m+\tfrac{1}{2})\mathscr{B}+\mathscr{B}=k\mathscr{B}.$$

Written in matrix form, these two equations become

$$\begin{bmatrix}-(k-\frac{1}{2}-m) & \sqrt{(l+\frac{1}{2})^2-m^2} \\ \sqrt{(l+\frac{1}{2})^2-m^2} & -(k-\frac{1}{2}+m)\end{bmatrix}\begin{bmatrix}\mathscr{A} \\ \mathscr{B}\end{bmatrix}=\begin{bmatrix}0 \\ 0\end{bmatrix}.$$

The necessary condition for a solution of this matrix equation to exist is that the determinant must be zero. This means that $(k-\frac{1}{2})^2=(l+\frac{1}{2})^2$. From this, we get two cases:

$$\text{Case 1: } k=l+1, \tag{8.109}$$

$$\text{Case 2: } k=-l. \tag{8.110}$$

For case 1,

$$\mathscr{B}=\frac{\sqrt{(l+\frac{1}{2})^2-m^2}}{l+\frac{1}{2}+m}\mathscr{A}=\frac{\sqrt{l+\frac{1}{2}-m}}{\sqrt{l+\frac{1}{2}+m}}\mathscr{A}.$$

We then have

$$\mathscr{Y}^{(+)}_{j,m}=\left(\sqrt{\frac{l+\frac{1}{2}+m}{2j}}\,Y_{l,m-\frac{1}{2}}+\sqrt{\frac{l+\frac{1}{2}-m}{2j}}\,Y_{l,m+\frac{1}{2}}\hat{\gamma}_{31}\right)\tfrac{1}{2}(I+\mathrm{i}\hat{\gamma}_{12})\mathscr{C}$$

where $\mathscr{C}$ is a Clifford number which is a function of r only. From Eqs. (8.101)

and (8.103), we have

$$|\boldsymbol{J}|^2 = |\boldsymbol{L}|^2 + \hat{\gamma}_0\hbar^2\boldsymbol{k} - \tfrac{1}{4}\hbar^2 I. \tag{8.111}$$

Thus

$$\begin{aligned}|\boldsymbol{J}|^2\mathscr{Y}^{(+)}_{j,m} &= \hbar^2[l(l+1) + (l+1) - \tfrac{1}{4}]\mathscr{Y}^{(+)}_{j,m}\\ &= \hbar^2(l+\tfrac{1}{2})(l+\tfrac{3}{2})\mathscr{Y}^{(+)}_{j,m}.\end{aligned}$$

If we insist that

$$|\boldsymbol{J}|^2\mathscr{Y}_{j,m} = \hbar^2 j(j+1)\mathscr{Y}_{j,m},$$

then

$$j = l + \tfrac{1}{2}$$

and we have

$$\mathscr{Y}^{(+)}_{j,m} = \left(\sqrt{\frac{j+m}{2j}}\,Y_{j-\frac{1}{2},m-\frac{1}{2}} + \sqrt{\frac{j-m}{2j}}\,Y_{j-\frac{1}{2},m+\frac{1}{2}}\hat{\gamma}_{31}\right)\tfrac{1}{2}(I + \mathrm{i}\hat{\gamma}_{12})\mathscr{C}$$

where $k = j + \frac{1}{2} = 1, 2, 3, \ldots$ and $m = -j, -j+1, \ldots, +j$.

A similar calculation for case 2 results in the identity:

$$\mathscr{Y}^{(-)}_{j,m} = \left(\sqrt{\frac{j+1-m}{2(j+1)}}\,Y_{j+\frac{1}{2},m-\frac{1}{2}} - \sqrt{\frac{j+1+m}{2(j+1)}}\,Y_{j+\frac{1}{2},m+\frac{1}{2}}\hat{\gamma}_{31}\right)\tfrac{1}{2}(I + \mathrm{i}\hat{\gamma}_{12})\mathscr{C}$$

where $k = -(j+\frac{1}{2}) = -1, -2, -3, \ldots$ and $m = -j, -j+1, \ldots, +j$.

In either case

$$J_3\mathscr{Y}^{(\pm)}_{j,m} = m\hbar\mathscr{Y}^{(\pm)}_{j,m}, \tag{8.112}$$

and

$$|\boldsymbol{J}|^2\mathscr{Y}^{(\pm)}_{j,m} = j(j+1)\hbar\mathscr{Y}^{(\pm)}_{j,m}. \tag{8.113}$$

Furthermore

$$\hat{\gamma}_0\boldsymbol{k}\mathscr{Y}^{(+)}_{j,m} = (j+\tfrac{1}{2})\mathscr{Y}^{(+)}_{j,m}, \tag{8.114}$$

and

$$\hat{\gamma}_0\boldsymbol{k}\mathscr{Y}^{(-)}_{j,m} = -(j+\tfrac{1}{2})\mathscr{Y}^{(-)}_{j,m}. \tag{8.115}$$

It is useful to require that these spherical harmonic Clifford functions also be eigenfunctions of $\hat{\gamma}_0$. To achieve this, we merely insert the term $\frac{1}{2}(I + \hat{\gamma}_0)$. That is

$$\mathscr{Y}^{(+)}_{j,m} = \tfrac{1}{4}\left(\sqrt{\frac{j+m}{2j}}\,Y_{j-\frac{1}{2},m-\frac{1}{2}} + \hat{\gamma}_{31}\sqrt{\frac{j-m}{2j}}\,Y_{j-\frac{1}{2},m+\frac{1}{2}}\right)(I + \mathrm{i}\hat{\gamma}_{12})(I + \hat{\gamma}_0)\mathscr{C} \tag{8.116}$$

and

$$\mathscr{Y}^{(-)}_{j,m} = \tfrac{1}{4}\left(\sqrt{\frac{j+1-m}{2(j+1)}}\, Y_{j+\frac{1}{2},m-\frac{1}{2}} - \hat{\gamma}_{31}\sqrt{\frac{j+1+m}{2(j+1)}}\, Y_{j+\frac{1}{2},m+\frac{1}{2}}\right)(I + \mathrm{i}\hat{\gamma}_{12})(I + \hat{\gamma}_0)\mathscr{C} \tag{8.117}$$

With this modification. Eqs. (8.112)–(8.115) remain valid. But we now have two more: namely:

$$k\mathscr{Y}^{(+)}_{j,m} = (j + \tfrac{1}{2})\mathscr{Y}^{(+)}_{j,m}, \tag{8.118}$$

and

$$k\mathscr{Y}^{(-)}_{j,m} = -(j + \tfrac{1}{2})\mathscr{Y}^{(-)}_{j,m}. \tag{8.119}$$

These spherical harmonic Clifford functions are now eigenfunctions of $\hat{\gamma}_0$ with the eigenvalue $+1$. One would think that it is also necessary to construct spherical harmonic functions with eigenvalue -1 for $\hat{\gamma}_0$. One way to do this is to insert either the 1-vector $\hat{\gamma}_r$ or the 2-vector $\hat{\gamma}_{r0}$ in front of $\mathscr{Y}^{(+)}_{j,m}$ and $\mathscr{Y}^{(-)}_{j,m}$ as defined by Eqs. (8.116) and (8.117). However, in either case the inserted factor can be absorbed into the nonangular component of the Clifford wave functions that are computed for Dirac's equation. Thus no generality is lost if one restricts consideration to the spherical harmonic Clifford functions defined by Eqs. (8.116) and (117).

It is interesting and perhaps useful to reorganize these spherical harmonic Clifford functions as rotation operators acting on the projection operator $\frac{1}{2}(I + \mathrm{i}\hat{\gamma}_{12})\frac{1}{2}(I + \hat{\gamma}_0)$. From Eqs. (8.97), (8.116), and (8.117), we have

$$\mathscr{Y}^{(+)}_{j,m} = \frac{\mathrm{e}^{\mathrm{i}(m-\frac{1}{2})\phi}}{4}\sqrt{\frac{j-m)!}{4\pi(j+m)!}} \times [(j+m)P^{m-\frac{1}{2}}_{j-\frac{1}{2}}(\cos\theta) + P^{m+\frac{1}{2}}_{j-\frac{1}{2}}(\cos\theta)\mathrm{e}^{\mathrm{i}\phi}\hat{\gamma}_{31}](I + \mathrm{i}\hat{\gamma}_{12})(I + \hat{\gamma}_0)\mathscr{C} \tag{8.120}$$

and

$$\mathscr{Y}^{(-)}_{j,m} = \frac{\mathrm{e}^{\mathrm{i}(m-\frac{1}{2})\phi}}{4}\sqrt{\frac{(j-m)!}{4\pi(j+m)!}} \times [(j-m+1)P^{m-\frac{1}{2}}_{j+\frac{1}{2}}(\cos\theta) - P^{m+\frac{1}{2}}_{j+\frac{1}{2}}(\cos\theta)\mathrm{e}^{\mathrm{i}\phi}\hat{\gamma}_{31}] \times (I + \mathrm{i}\hat{\gamma}_{12})(I + \hat{\gamma}_0)\mathscr{C}. \tag{8.121}$$

It is possible to replace $\mathrm{e}^{\mathrm{i}\phi}\hat{\gamma}_{31}$ in both Eq. (8.120) and (8.121) by a real Clifford number. To accomplish this we note that $\mathrm{i}\hat{\gamma}_{12}(I + \mathrm{i}\hat{\gamma}_{12}) = (I + \mathrm{i}\hat{\gamma}_{12})$.

Therefore:

$$\begin{aligned} e^{i\phi}\hat{\gamma}_{31}(I + i\hat{\gamma}_{12}) &= (\hat{\gamma}_{31}\cos\phi + i\hat{\gamma}_{31}\sin\phi)(I + i\hat{\gamma}_{12}) \\ &= (\hat{\gamma}_{31}\cos\phi - (i\hat{\gamma}_{31})(i\hat{\gamma}_{12})\sin\phi)(I + i\hat{\gamma}_{12}) \\ &= (\hat{\gamma}_{31}\cos\phi - \hat{\gamma}_{23}\sin\phi)(I + i\hat{\gamma}_{12}). \end{aligned}$$

Interestingly enough, it turns out that if one uses spherical coordinates,

$$\hat{\gamma}_{31}\cos\phi - \hat{\gamma}_{23}\sin\phi = \frac{\hat{\gamma}_{r\theta}}{r}. \tag{8.122}$$

The 2-vector $\hat{\gamma}_{r\theta}/r$ is normalized so that $(\hat{\gamma}_{r\theta}/r)^2 = -I$.

Equations (8.120) and (8.121) now become

$$\mathscr{Y}^{(+)}_{j,m} = e^{i(m-\frac{1}{2})\phi}\sqrt{\frac{(j-m)!}{4\pi(j+m)!}} \times \left[(j+m)P^{m-\frac{1}{2}}_{j-\frac{1}{2}}(\cos\theta) + \frac{\hat{\gamma}_{r\theta}}{r}P^{m+\frac{1}{2}}_{j-\frac{1}{2}}(\cos\theta)\right]\tfrac{1}{4}(I + i\hat{\gamma}_{12})(I + \hat{\gamma}_0)\mathscr{C} \tag{8.123}$$

and

$$\mathscr{Y}^{(-)}_{j,m} = e^{i(m-\frac{1}{2})\phi}\sqrt{\frac{(j-m)!}{4\pi(j+m)!}} \times \left[(j-m+1)P^{m-\frac{1}{2}}_{j+\frac{1}{2}}(\cos\theta) - \frac{\hat{\gamma}_{r\theta}}{r}P^{m+\frac{1}{2}}_{j+\frac{1}{2}}(\cos\theta)\right]\tfrac{1}{4}(I + i\hat{\gamma}_{12})(I + \hat{\gamma}_0)\mathscr{C} \tag{8.124}$$

It is now possible to interpret these spherical harmonic Clifford functions as the product of a scalar function, a rotation operator, and the projection operator $\frac{1}{4}(I + i\hat{\gamma}_{12})(I + \hat{\gamma}_0)$. Suppose we define

$$(N^m_j(\theta))^{\frac{1}{2}}\cos\frac{\beta}{2} = \sqrt{\frac{(j-m)!}{4\pi(j+m)!}}(j+m)P^{m-\frac{1}{2}}_{j-\frac{1}{2}}(\cos\theta) \tag{8.125}$$

and

$$(N^m_j(\theta))^{\frac{1}{2}}\sin\frac{\beta}{2} = \sqrt{\frac{(j-m)!}{4\pi(j+m)!}}P^{m+\frac{1}{2}}_{j-\frac{1}{2}}(\cos\theta) \tag{8.126}$$

Then Eq. (8.123) becomes

$$\mathscr{Y}^{(+)}_{j,m} = e^{i(m-\frac{1}{2})\phi}(N^m_j(\theta))^{\frac{1}{2}}\left[I\cos\frac{\beta}{2} + \frac{\hat{\gamma}_{r\theta}}{r}\sin\frac{\beta}{2}\right]\tfrac{1}{4}(I + i\hat{\gamma}_{12})(I + \hat{\gamma}_0)\mathscr{C}. \tag{8.127}$$

The angle β depends not only on θ but also on the indices j and m. Thus it is perhaps true that we should attach the indices j and m to the angle β. However, doing so becomes somewhat cumbersome. Furthermore the angle β will not appear in any expression without the indices j and m appearing elsewhere in that same expression. Thus the addition of the indices to the angle β seems unnecessary.

If we square both sides of Eqs. (8.125) and (8.126), we get

$$N_j^m(\theta) = \frac{1}{4\pi}\frac{(j-m)!}{(j+m)!}\left[(j+m)^2(P_{j-\frac{1}{2}}^{m-\frac{1}{2}})^2 + (P_{j-\frac{1}{2}}^{m+\frac{1}{2}})^2\right]. \tag{8.128}$$

The $N_j^m(\theta)$ functions have been normalized so that

$$\int_0^\pi \int_0^{2\pi} N_j^m(\theta)\, d\phi \sin\theta\, d\theta = 2\pi \int_0^\pi N_j^m(\theta) \sin\theta\, d\theta = 1. \tag{8.129}$$

It is somewhat surprising that $\mathscr{Y}_{j,m}^{(-)}$ can also be expressed in terms of the same angle β. Accordiing to Magnus, Oberhettinger, and Soni (1966, p. 171), we have:

$$(p-q+1)P_{p+1}^q(\cos\theta) = (p+q+1)\cos\theta P_p^q(\cos\theta) + \sin\theta P_p^{q+1}(\cos\theta) \tag{8.130}$$

and

$$-P_{p+1}^q(\cos\theta) = (p+q)\sin\theta P_p^{q-1}(\cos\theta) - \cos\theta P_p^q(\cos\theta). \tag{8.131}$$

Equation (8.130) implies that

$$(j-m+1)P_{j+\frac{1}{2}}^{m-\frac{1}{2}} = (j+m)\cos\theta P_{j-\frac{1}{2}}^{m-\frac{1}{2}} + \sin\theta P_{j-\frac{1}{2}}^{m+\frac{1}{2}} \tag{8.132}$$

or

$$\sqrt{\frac{(j-m)!}{4\pi(j+m)!}}(j-m+1)P_{j+\frac{1}{2}}^{m-\frac{1}{2}}$$
$$= (N_j^m)^{\frac{1}{2}}\left(\cos\theta\cos\frac{\beta}{2} + \sin\theta\sin\frac{\beta}{2}\right) = (N_j^m)^{\frac{1}{2}}\cos\left(\theta - \frac{\beta}{2}\right). \tag{8.133}$$

In a similar fashion, Eq. (8.131) implies that

$$-\sqrt{\frac{(j-m)!}{4\pi(j+m)!}}P_{j+\frac{1}{2}}^{m+\frac{1}{2}} = (N_j^m)^{\frac{1}{2}}\sin\left(\theta - \frac{\beta}{2}\right). \tag{8.134}$$

With these relations, Eq. (8.124) becomes

$$\mathscr{Y}^{(-)}_{j,m} = e^{i(m-\frac{1}{2})\phi}(N^m_j(\theta))^{\frac{1}{2}} \times \left[I\cos\left(\theta - \frac{\beta}{2}\right) + \frac{\hat{\gamma}_{r\theta}}{r}\sin\left(\theta - \frac{\beta}{2}\right)\right](I + i\hat{\gamma}_{12})(I + \hat{\gamma}_0)\mathscr{C}. \quad (8.135)$$

From the computations of Section 8.1 it appears that most if not all expectation values can be extracted from the product $\psi\psi^\dagger$. With this motivation, let us compute the product $\mathscr{Y}^{(+)}_{j,m}\mathscr{Y}^{(+)\dagger}_{j,m}$. To do this, we will assume that the constant Clifford number $\mathscr{C}$ that appears in Eq. (8.127) is c-unitary. We will also use the fact that the square of the projection operator $\frac{1}{4}(I + i\hat{\gamma}_{12})(I + \hat{\gamma}_0)$ is equal to itself. We then have

$$\mathscr{Y}^{(+)}_{j,m}\mathscr{Y}^{(+)\dagger}_{j,m} = \tfrac{1}{4}N^m_j\left(I\cos\frac{\beta}{2} + \frac{\hat{\gamma}_{r\theta}}{r}\sin\frac{\beta}{2}\right) \times (I + i\hat{\gamma}_{12})(I + \hat{\gamma}_0)\left(I\cos\frac{\beta}{2} - \frac{\hat{\gamma}_{r\theta}}{r}\sin\frac{\beta}{2}\right). \quad (8.136)$$

The operator $\hat{\gamma}_0$ commutes with the rotation operator and

$$\left(I\cos\frac{\beta}{2} + \frac{\hat{\gamma}_{r\theta}}{r}\sin\frac{\beta}{2}\right)\left(I\cos\frac{\beta}{2} - \frac{\hat{\gamma}_{r\theta}}{r}\sin\frac{\beta}{2}\right) = I.$$

From Eq. (8.122), $\hat{\gamma}_{12}$ anticommutes with $\hat{\gamma}_{r\theta}$. Thus

$$\left(I\cos\frac{\beta}{2} + \frac{\hat{\gamma}_{r\theta}}{r}\sin\frac{\beta}{2}\right)(i\hat{\gamma}_{12})\left(I\cos\frac{\beta}{2} - \frac{\hat{\gamma}_{r\theta}}{r}\sin\frac{\beta}{2}\right)$$
$$= (i\hat{\gamma}_{12})\left(I\cos\frac{\beta}{2} - \frac{\hat{\gamma}_{r\theta}}{r}\sin\frac{\beta}{2}\right)^2 = (i\hat{\gamma}_{12})\left(I\cos\beta - \frac{\hat{\gamma}_{r\theta}}{r}\sin\beta\right).$$

Thus Eq. (8.136) becomes

$$\mathscr{Y}^{(+)}_{j,m}\mathscr{Y}^{(+)\dagger}_{j,m} = \tfrac{1}{4}N^m_j(I + \hat{\gamma}_0)\left[I + i\hat{\gamma}_{12}\left(I\cos\beta - \frac{\hat{\gamma}_{r\theta}}{r}\sin\beta\right)\right]. \quad (8.137)$$

It can also be shown that

$$\hat{\gamma}_{12} = \frac{\hat{\gamma}_{\theta\phi}}{r^2\sin\theta}\cos\theta - \frac{\hat{\gamma}_{\phi r}}{r\sin\theta}\sin\theta. \quad (8.138)$$

If one substitutes the right-hand side of Eq. (8.138) into Eq. (8.137) and then

carries out the required multiplications, one finally obtains

$$\mathscr{Y}^{(+)}_{j,m}\mathscr{Y}^{(+)\dagger}_{j,m} = \tfrac{1}{4}N^m_j(\theta)\left[I + \frac{i\gamma_{\theta\phi}}{r^2\sin\theta}\cos(\beta-\theta) + \frac{i\hat{\gamma}_{\phi r}}{r\sin\theta}\sin(\beta-\theta)\right][I+\gamma_0]. \tag{8.139}$$

To obtain the product $\mathscr{Y}^{(-)}_{j,m}\mathscr{Y}^{(-)\dagger}_{j,m}$ it is only necessary to replace the angle $\beta/2$ that appears in Eq. (8.127) by the angle $(\theta - \beta/2)$ that appears in Eq. (8.135). This means that instead of getting $\beta - \theta$ or $2(\beta/2) - \theta$ in Eq. (8.139), one gets $2(\theta - \beta/2) - \theta$ or $\theta - \beta$. Since $\cos(\theta - \beta) = \cos(\beta - \theta)$ and $\sin(\theta - \beta) = -\sin(\beta - \theta)$, we get:

$$\mathscr{Y}^{(-)}_{j,m}\mathscr{Y}^{(-)\dagger}_{j,m} = \tfrac{1}{4}N^m_j(\theta)\left[I + \frac{i\hat{\gamma}_{\theta\phi}}{r^2\sin\theta}\cos(\beta-\theta) - \frac{i\hat{\gamma}_{\phi r}}{r\sin\theta}\sin(\beta-\theta)\right](I+\hat{\gamma}_0). \tag{8.140}$$

Because of the critical role that these products of spherical harmonic Clifford functions play in the computation of currents and dipoles, it is useful to introduce an explicit notation for the functions that appear in Eqs. (8.139) and (8.140). In particular, let us define:

$$C^m_j(\kappa) = N^m_j(\theta)\cos(\beta - \theta) \tag{8.141}$$

and

$$S^m_j(\theta) = N^m_j(\theta)\sin(\beta - \theta). \tag{8.142}$$

With this notation, we have

$$\mathscr{Y}^{(+)}_{j,m}\mathscr{Y}^{(+)\dagger}_{j,m} = \frac{1}{4}\left[IN^m_j(\theta) + \frac{i\hat{\gamma}_{\theta\phi}}{r^2\sin\theta}C^m_j(\theta) + \frac{i\hat{\gamma}_{\phi r}}{r\sin\theta}S^m_j(\theta)\right](I+\hat{\gamma}_0) \tag{8.143}$$

and

$$\mathscr{Y}^{(-)}_{j,m}\mathscr{Y}^{(-)\dagger}_{j,m} = \frac{1}{4}\left[IN^m_j(\theta) + \frac{i\hat{\gamma}_{\theta\phi}}{r^2\sin\theta}C^m_j(\theta) - \frac{i\hat{\gamma}_{\phi r}}{r\sin\theta}S^m_j(\theta)\right](I+\hat{\gamma}_0) \tag{8.144}$$

To obtain explicit formulas for $C^m_j(\theta)$ and $S^m_j(\theta)$ in terms of associated Legendre functions, one should note that

$$\begin{aligned}\cos(\beta - \theta) &= \cos\left(\frac{\beta}{2} - \left(\theta - \frac{\beta}{2}\right)\right)\\ &= \cos\frac{\beta}{2}\cos\left(\theta - \frac{\beta}{2}\right) + \sin\frac{\beta}{2}\sin\left(\theta - \frac{\beta}{2}\right).\end{aligned}$$

Using this relation along with Eqs. (8.125), (8.126), (8.133), and (8.134), one gets

$$\begin{aligned} C_j^m(\theta) &= N_j^m(\theta)\cos(\beta - \theta) \\ &= \frac{1}{4\pi}\frac{(j-m)!}{(j+m)!}\left[(j+m)(j-m+1)P_{j-\frac{1}{2}}^{m-\frac{1}{2}}P_{j+\frac{1}{2}}^{m-\frac{1}{2}} - P_{j-\frac{1}{2}}^{m+\frac{1}{2}}P_{j+\frac{1}{2}}^{m+\frac{1}{2}}\right]. \end{aligned} \tag{8.145}$$

In a similar fashion, one finds that

$$\begin{aligned} S_j^m(\theta) &= N_j^m(\theta)\sin(\beta - \theta) \\ &= \frac{1}{4\pi}\frac{(j-m)!}{(j+m)!}\left[(j-m+1)P_{j-\frac{1}{2}}^{m+\frac{1}{2}}P_{j+\frac{1}{2}}^{m-\frac{1}{2}} + (j+m)P_{j-\frac{1}{2}}^{m-\frac{1}{2}}P_{j+\frac{1}{2}}^{m+\frac{1}{2}}\right]. \end{aligned} \tag{8.146}$$

From Eqs. (8.128), (8.145), and (8.146), it is possible to show that

$$N_j^{-m}(\theta) = N_j^m(\theta), \tag{8.147}$$

$$C_j^{-m}(\theta) = -C_j^m(\theta). \tag{8.148}$$

$$S_j^{-m}(\theta) = -S_j^m(\theta). \tag{8.149}$$

(See Problem 8.23.)

From Eqs. (8.98), (8.145), and (8.146), it is also possible to show that the $C_j^m(\theta)$'s are polynomials in odd powers of $\cos\theta$ and the $S_j^m(\theta)$'s are polynomials in odd powers of $\sin\theta$. For low values of j and m, we have

$$C_{\frac{1}{2}}^{\frac{1}{2}}(\theta) = \frac{1}{4\pi}\cos\theta$$

$$S_{\frac{1}{2}}^{\frac{1}{2}}(\theta) = -\frac{1}{4\pi}\sin\theta$$

$$N_{\frac{1}{2}}^{\frac{1}{2}}(\theta) = \frac{1}{4\pi}$$

$$C_{3/2}^{3/2}(\theta) = \frac{1}{8\pi}\sin^2\theta\cos\theta$$

$$S_{3/2}^{3/2}(\theta) = -\frac{1}{8\pi}\sin^3\theta$$

$$N_{3/2}^{3/2}(\theta) = \frac{1}{8\pi}\sin^2\theta$$

$$C^{1/2}_{3/2}(\theta) = \frac{1}{8\pi}(-9\sin^2\theta + 4)\cos\theta$$

$$S^{1/2}_{3/2}(\theta) = \frac{1}{8\pi}(9\sin^2\theta - 8)\sin\theta$$

$$N^{1/2}_{3/2}(\theta) = \frac{1}{8\pi}(-3\sin^2\theta + 4)$$

$$C^{5/2}_{5/2}(\theta) = \frac{15}{32\pi}\sin^4\theta\cos\theta$$

$$S^{5/2}_{5/2}(\theta) = -\frac{15}{32\pi}\sin^5\theta$$

$$N^{5/2}_{5/2}(\theta) = \frac{15}{32\pi}\sin^4\theta$$

$$C^{3/2}_{5/2}(\theta) = \frac{3}{32\pi}(-25\sin^4\theta + 16\sin^2\theta)\cos\theta$$

$$S^{3/2}_{5/2}(\theta) = \frac{3}{32\pi}(25\sin^4\theta - 24\sin^2\theta)\sin\theta$$

$$N^{3/2}_{5/2}(\theta) = \frac{3}{32\pi}(-15\sin^4\theta + 16\sin^2\theta)$$

$$C^{1/2}_{5/2}(\theta) = \frac{3}{16\pi}(25\sin^4\theta - 24\sin^2\theta + 4)\cos\theta$$

$$S^{1/2}_{5/2}(\theta) = \frac{3}{16\pi}(-25\sin^4\theta + 36\sin^2\theta - 12)\sin\theta$$

$$N^{1/2}_{5/2}(\theta) = \frac{3}{16\pi}(5\sin^4\theta - 8\sin^2\theta + 4)$$

Problem 8.17. Show that

(1) $$\left[\boldsymbol{L}_3, \sum_{k=1}^{3} \gamma^k \frac{\partial}{\partial x^k}\right] = i\hbar\left(\gamma^1 \frac{\partial}{\partial x^2} - \gamma^2 \frac{\partial}{\partial x^1}\right).$$

(It is understood here that the square brackets designate the commutator brackets. That is $[\mathscr{A}, \mathscr{B}] = \mathscr{A}\mathscr{B} - \mathscr{B}\mathscr{A}$.)

(2) Show that $\left[\boldsymbol{J}_3, \sum_{k=1}^{3} \gamma^k \frac{\partial}{\partial x^k}\right] = 0.$

(3) Show that $[J_3, f(r)\} = 0$.
(Comment: parts (2) and (3) serve to verify Eq. (8.88).)

Problem 8.18. Show that the operator $\boldsymbol{k}$ commutes with $\hat{\gamma}_r$, r^2, $\hat{\gamma}_0$, $|\boldsymbol{J}|^2$, $\boldsymbol{J}_1$, $\boldsymbol{J}_2$, and $\boldsymbol{J}_3$.

Problem 8.19. Show that

$$(\boldsymbol{k})^2 = \frac{1}{\hbar^2}|\boldsymbol{J}|^2 + \tfrac{1}{4}I.$$

Problem 8.20. Use Eqs. (8.97) and (8.98) to show that

$$\boldsymbol{L}_+ Y_{p,q} = \hbar\sqrt{(p-q)(p+q+1)}\, Y_{p,q+1}.$$

(Suggestion: first show that

$$\boldsymbol{L}_+ = \hbar\, \mathrm{e}^{\mathrm{i}\phi}\left(\frac{\partial}{\partial\theta} + \mathrm{i}\frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\phi}\right).)$$

Also show that from Eq. (8.98)

$$P_p^q(\cos\theta) = \frac{1}{2^p p!}\sin^q\theta\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right)^{p+q}\cos^{2p}\theta.$$

If you wish to show that $\boldsymbol{L}_- Y_{p,q} = \hbar\sqrt{(p+q)(p-q+1)}\, Y_{p,q-1}$, it is probably useful to use the relation that

$$P_p^{-q} = (-1)^q\frac{(p-q)!}{(p+q)!}P_p^q.$$

Problem 8.21.

(1) Show that if we apply the charge conjugate operator defined in Prob. 8.6 to the operator $\boldsymbol{k}$ of Eq. (8.104), one gets $\boldsymbol{k}^{\mathrm{C}} = -\boldsymbol{k}$. Use this result to show that

$$\boldsymbol{k}(\mathscr{Y}_{j,m}^{(\pm)})^{\mathrm{C}} = \mp(j+\tfrac{1}{2})(\mathscr{Y}_{j,m}^{(\pm)})^{\mathrm{C}}.$$

(2) Show that $J_3(\mathscr{Y}_{j,m}^{(\pm)})^{\mathrm{C}} = -m\hbar(\mathscr{Y}_{j,m}^{(\pm)})^{\mathrm{C}}$ and

$$|\boldsymbol{J}|^2(\mathscr{Y}_{j,m}^{(\pm)})^{\mathrm{C}} = \hbar^2 j(j+1)(\mathscr{Y}_{j,m}^{(\pm)})^{\mathrm{C}}.$$

Problem 8.22. Use the relation that

$$\int_{-1}^{1} P_p^q(x) P_r^q(x)\,\mathrm{d}x = 0 \quad \text{if } p \neq r$$

and Eq. (8.145) to show that

$$\int_{0}^{\pi} C_j^m(\theta) \sin\theta\,\mathrm{d}\theta = 0.$$

Problem 8.23. Use the relation that

$$P_p^{-q} = (-1)^q \frac{(p-q)!}{(p+q)!} P_p^q$$

and Eqs. (8.128), (8.145), and (8.146) to show that $N_j^{-m}(\theta) = N_j^m(\theta)$, $C_j^{-m}(\theta) = -C_j^m(\theta)$, and $S_j^{-m}(\theta) = -S_j^m(\theta)$.

Problem 8.24. Define the 2-vector

$$\frac{\mathrm{i}\hat{\gamma}_{\theta\phi}}{r^2 \sin^2\theta} = \mathrm{i}\hat{\gamma}_{23}\frac{x^1}{r} + \mathrm{i}\hat{\gamma}_{31}\frac{x^2}{r} + \mathrm{i}\hat{\gamma}_{12}\frac{x^3}{r}.$$

Show

$$\left(\frac{\mathrm{i}\hat{\gamma}_{\theta\phi}}{r^2 \sin^2\theta}\right)\mathcal{Y}_{j,m}^{(-)} = \mathcal{Y}_{j,m}^{(+)}.$$

(Comment: this shows that $\mathcal{Y}_{j,m}^{(-)}$ and $\mathcal{Y}_{j,m}^{(+)}$ have opposite parity.)

Problem 8.25. Under the symmetry operator of parity $\boldsymbol{P}$; $\boldsymbol{P}x^k = -x^k$ for $k = 1, 2$, and 3; and $\boldsymbol{P}f(\theta, \phi) = f(\pi - \theta, \phi + \pi)$.

(1) Use Eqs. (8.97) and (8.98) to show

$$\boldsymbol{P}Y_{p,q}(\theta, \phi) = (-1)^p Y_{p,q}(\theta, \phi).$$

(2) Use the result of part (1) to show

$$\boldsymbol{P}\mathcal{Y}_{j,m}^{(+)} = (-1)^{j-\frac{1}{2}}\mathcal{Y}_{j,m}^{(+)} = (-1)^l \mathcal{Y}_{j,m}^{(+)}$$

and

$$\boldsymbol{P}\mathcal{Y}_{j,m}^{(-)} = (-1)^{j+\frac{1}{2}}\mathcal{Y}_{j,m}^{(-)} = (-1)^l \mathcal{Y}_{j,m}^{(-)}$$

8.5 Clifford Solutions of Dirac's Equation for Hydrogen-like Atoms

Having computed the Clifford eigenfunctions of the operator $\boldsymbol{k}$, we now turn to the problem of finding the Clifford function solutions to Dirac's equation for hydrogen-like atoms.

Using Eq. (8.103) and the relation

$$\hat{\gamma}^r = \sum_{k=1}^{3} \hat{\gamma}^k \frac{x^k}{r},$$

it is not difficult to show that

$$\sum_{k=1}^{3} \hat{\gamma}^k \frac{\partial}{\partial x^k} = \hat{\gamma}^r\left(\frac{1}{r}\frac{\partial}{\partial r} r\right) - \frac{1}{r}\hat{\gamma}^{r0}\boldsymbol{k}. \tag{8.150}$$

From Eqs. (8.150) and (8.85), Dirac's equation for an electron in a Coulomb potential with energy E becomes

$$\left[\hat{\gamma}^0\left(E + \frac{Ze^2}{4\pi r}\right) - mc^2 I + \mathrm{i}\hbar c\hat{\gamma}^r\left(\frac{1}{r}\frac{\partial}{\partial r} r\right) - \frac{\mathrm{i}\hbar c}{r}\hat{\gamma}^{r0}\boldsymbol{k}\right]\psi = 0. \tag{8.151}$$

To solve Eq. (8.151), it might be thought that we should consider Clifford functions of the form:

$$\psi = [A(r) + \hat{\gamma}^0 B(r) + \mathrm{i}\hat{\gamma}^r C(r) + \mathrm{i}\hat{\gamma}^{r0} D(r)]\mathscr{Y}^{(\pm)}_{j,m}. \tag{8.152}$$

However, since $\hat{\gamma}^0\mathscr{Y}^{(\pm)}_{j,m} = \mathscr{Y}^{(\pm)}_{j,m}$, Eq. (8.152) may be rewritten as

$$\psi = [I(A(r) + B(r)) + \mathrm{i}\hat{\gamma}^r(C(r) + D(r))]\mathscr{Y}^{(\pm)}_{j,m}.$$

This form can obviously be simplified. If we replace $(A(r) + B(r))$ by $F(r)/r$ and $(C(r) + D(r))$ by $G(r)/r$, we have

$$\psi = \left[I\frac{F(r)}{r} + \mathrm{i}\hat{\gamma}^r\frac{G(r)}{r}\right]\mathscr{Y}^{(\pm)}_{j,m}. \tag{8.153}$$

If one now substitutes the form for ψ that appears in Eq. (8.153) into Eq. (8.151), and absorbs the matrix $\hat{\gamma}^0$ into $\mathscr{Y}^{(\pm)}_{j,m}$ wherever it occurs, we have

$$\left[\left(E + \frac{Ze^2}{4\pi r} - mc^2\right)\frac{F(r)}{r} + \frac{\hbar c}{r}\left(\frac{\partial}{\partial r} + \frac{k}{r}\right)G(r)\right]\mathscr{Y}^{(\pm)}_{j,m}$$
$$-\mathrm{i}\hat{\gamma}^r\left[\left(E + \frac{Ze^2}{4\pi r} + mc^2\right)\frac{G(r)}{r} - \frac{\hbar c}{r}\left(\frac{\partial}{\partial r} - \frac{k}{r}\right)F(r)\right]\mathscr{Y}^{(\pm)}_{j,m} = 0.$$

Therefore

$$\left(E + \frac{Ze^2}{4\pi r} - mc^2\right)\frac{F(r)}{r} + \frac{\hbar c}{r}\left(\frac{\partial}{\partial r} + \frac{k}{r}\right)G(r) = 0, \tag{8.154}$$

and

$$\left(E + \frac{Ze^2}{4\pi r} + mc^2\right)\frac{G(r)}{r} - \frac{\hbar c}{r}\left(\frac{\partial}{\partial r} - \frac{k}{r}\right)F(r) = 0. \tag{8.155}$$

These same two equations appear in the usual spinor formalism used to deal with Dirac's equation for hydrogen-like atoms. To solve them, it is useful to abbreviate the constants that appear in the two equations. In particular let:

$$\alpha = \frac{e^2}{4\pi\hbar c}, \tag{8.156}$$

$$\alpha_+ = \frac{mc^2 + E}{\hbar c}, \tag{8.157}$$

$$\alpha_- = \frac{mc^2 - E}{\hbar c}. \tag{8.158}$$

(The constant α is of course the *fine structure constant.*)

With these abbreviations, Eqs. (8.154) and (8.155) become

$$\frac{\mathrm{d}G}{\mathrm{d}r} + \frac{k}{r}G - (\alpha_-)F = -\frac{Z\alpha}{r}F \tag{8.159}$$

and

$$\frac{\mathrm{d}F}{\mathrm{d}r} - \frac{k}{r}F - (\alpha_+)G = \frac{Z\alpha}{r}G. \tag{8.160}$$

If we consider asymptotic solutions of Eqs. (8.159) and (8.160), we have

$$\frac{\mathrm{d}G}{\mathrm{d}r} - (\alpha_-)F \sim 0 \qquad \text{and} \qquad \frac{\mathrm{d}F}{\mathrm{d}r} - (\alpha_+)G \sim 0.$$

The most general solution to these two equations is

$$F \sim a(\alpha_+)^{\frac{1}{2}}\,\mathrm{e}^{\lambda r} + b(\alpha_+)^{\frac{1}{2}}\,\mathrm{e}^{-\lambda r}$$

and

$$G \sim a(\alpha_-)^{\frac{1}{2}}\,\mathrm{e}^{\lambda r} - b(\alpha_-)^{\frac{1}{2}}\,\mathrm{e}^{-\lambda r}$$

where

$$\lambda = (\alpha_+\alpha_-)^{\frac{1}{2}}.$$

If we insist that our solutions remain bounded for large values of r then $a = 0$ and we have

$$F \sim b(\alpha_+)^{\frac{1}{2}} \, \mathrm{e}^{-\lambda r} \tag{8.161}$$

and

$$G \sim -b(\alpha_-)^{\frac{1}{2}} \, \mathrm{e}^{-\lambda r}. \tag{8.162}$$

On the other hand, suppose we consider solutions of Eqs. (8.159) and (8.160) near the origin. Let

$$F = r^s \sum_{j=0}^{\infty} f_j r^j \qquad \text{and} \qquad G = r^s \sum_{j=0}^{\infty} g_j r^j.$$

If we plug these infinite series into Eqs. (8.159) and (8.160) and then equate the coefficients of r^{s-1}, we get

$$(s + k)g_0 = -Z\alpha f_0 \qquad \text{and} \qquad (s - k)f_0 = Z\alpha g_0.$$

Therefore $(s^2 - k^2)f_0 g_0 = -Z^2\alpha^2 f_0 g_0$ or

$$s = \pm\sqrt{k^2 - Z^2\alpha^2}.$$

If we require that our solutions be reasonably well behaved near the origin, then we must insist that s be positive, that is

$$s = +\sqrt{k^2 - Z^2\alpha^2}. \tag{8.163}$$

To obtain exact solutions of Eqs. (8.159) and (8.160) in terms of functions which have been studied in detail, one must carry out some manipulations. To do this, we first multiply Eq. (8.159) by $(\alpha_+)^{\frac{1}{2}}$ and Eq. (8.160) by $(\alpha_-)^{\frac{1}{2}}$. We then add and subtract the two resulting equations. After introducing some new functions, the result is

$$\frac{\mathrm{d}F_1}{\mathrm{d}r} + (\alpha_+\alpha_-)^{\frac{1}{2}}F_1 - \frac{Z\alpha(\alpha_+ - \alpha_-)}{2(\alpha_+\alpha_-)^{\frac{1}{2}}}\frac{F_1}{r} = \left[k + \frac{Z\alpha(\alpha_+ + \alpha_-)}{2(\alpha_+\alpha_-)^{\frac{1}{2}}}\right]\frac{F_2}{r} \tag{8.164}$$

and

$$\frac{\mathrm{d}F_2}{\mathrm{d}r} - (\alpha_+\alpha_-)^{\frac{1}{2}}F_2 + \frac{Z\alpha(\alpha_+ - \alpha_-)}{2(\alpha_+\alpha_-)^{\frac{1}{2}}}\frac{F_2}{r} = \left[k - \frac{Z\alpha(\alpha_+ + \alpha_-)}{2(\alpha_+\alpha_-)^{\frac{1}{2}}}\right]\frac{F_1}{r} \tag{8.165}$$

where

$$F_1(r) = (\alpha_-)^{\frac{1}{2}}F(r) - (\alpha_+)^{\frac{1}{2}}G(r) \tag{8.166}$$

and

$$F_2(r) = (\alpha_-)^{\frac{1}{2}}F(r) + (\alpha_+)^{\frac{1}{2}}G(r). \tag{8.167}$$

Suppose we define

$$A = \frac{Z\alpha(\alpha_+ - \alpha_-)}{2(\alpha_+\alpha_-)^{\frac{1}{2}}} = \frac{Z\alpha E}{\sqrt{m^2c^4 - E^2}}, \tag{8.168}$$

$$B_+ = k + \frac{Z\alpha(\alpha_+ + \alpha_1)}{2(\alpha_+\alpha_-)^{\frac{1}{2}}} = k + \frac{Z\alpha mc^2}{\sqrt{m^2c^4 - E^2}}, \tag{8.169}$$

$$B_- = k - \frac{Z\alpha(\alpha_+ + \alpha_-)}{2(\alpha_+\alpha_-)^{\frac{1}{2}}} = k - \frac{Z\alpha mc^2}{\sqrt{m^2c^4 - E^2}}, \tag{8.170}$$

and

$$\lambda = (\alpha_+\alpha_-)^{\frac{1}{2}}. \tag{8.171}$$

Then Eqs. (8.164) and (8.165) become

$$r^A\,\mathrm{e}^{-\lambda r}\frac{\mathrm{d}}{\mathrm{d}r}(r^{-A}\,\mathrm{e}^{\lambda r}F_1) = \frac{B_+}{r}F_2, \tag{8.172}$$

and

$$r^{-A}\,\mathrm{e}^{\lambda r}\frac{\mathrm{d}}{\mathrm{d}r}(r^A\,\mathrm{e}^{-\lambda r}F_2) = \frac{B_-}{r}F_1. \tag{8.173}$$

Having examined the form of solutions near $r = \infty$ and near $r = 0$, it now appears to be useful to substitute

$$F_1(r) = c_1\rho^s\,\mathrm{e}^{-(\rho/2)}f_1(\rho) \tag{8.174}$$

and

$$F_2(r) = c_2\rho^s\,\mathrm{e}^{-(\rho/2)}f_2(\rho), \tag{8.175}$$

where $\rho = 2\lambda r$ and s will assume some convenient value. With these substitutions, Eqs. (8.172) and (8.173) become

$$\frac{c_1}{B_+c_2}\rho^{A-s+1}\frac{\mathrm{d}}{\mathrm{d}\rho}(\rho^{-A+s}f_1(\rho)) = f_2(\rho) \tag{8.176}$$

and

$$\frac{c_2}{B_-c_1}\rho^{-A-s+1}\,\mathrm{e}^{\rho}\frac{\mathrm{d}}{\mathrm{d}\rho}(\rho^{A+s}\,\mathrm{e}^{-\rho}f_2(\rho)) = f_1(\rho). \tag{8.177}$$

If we substitute the left-hand side of Eq. (8.176) into Eq. (8.177) and carry out the obvious calculations, we get

$$\rho\frac{\mathrm{d}^2 f_1}{\mathrm{d}\rho^2} + [(2s+1)-\rho]\frac{\mathrm{d}f_1}{\mathrm{d}\rho} - (s-A)f_1 = (A^2 + B_+B_- - s^2)\frac{f_1}{\rho}. \tag{8.178}$$

A similar calculation for f_2, gives us

$$\rho\frac{\mathrm{d}^2 f_2}{\mathrm{d}\rho^2} + [(2s+1)-\rho]\frac{\mathrm{d}f_2}{\mathrm{d}\rho} - (s-A+1)f_2 = (A^2 + B_+B_- - s^2)\frac{f_2}{\rho}. \tag{8.179}$$

It now becomes convenient to adjust s so that

$$s = (A^2 + B_+B_-)^{\frac{1}{2}} = (k^2 - Z^2\alpha^2)^{\frac{1}{2}}. \tag{8.180}$$

(Of course this is the same value for s that was obtained in Eq. (8.163).)

Our two equations then become

$$\rho\frac{\mathrm{d}^2 f_1}{\mathrm{d}\rho^2} + [(2s+1)-\rho]\frac{\mathrm{d}f_1}{\mathrm{d}\rho} - (s-A)f_1 = 0. \tag{8.181}$$

and

$$\rho\frac{\mathrm{d}^2 f_2}{\mathrm{d}\rho^2} + [(2s+1)-\rho]\frac{\mathrm{d}f_2}{\mathrm{d}\rho} - (s-A+1)f_2 = 0. \tag{8.182}$$

These two equations are known as special examples of *Kummer's differential equation*. In its general form, Kummer's differential equation is written as

$$z\frac{\mathrm{d}^2 w}{\mathrm{d}z^2} + (c-z)\frac{\mathrm{d}w}{\mathrm{d}z} - aw = 0. \tag{8.183}$$

Not suprisingly, one of the solutions to this equation is known as *Kummer's function*. It is also known as a *confluent hypergeometric function*. In particular this solution is denoted by ${}_1F_1(a; c; z)$, that is

$$w_1(z) = {}_1F_1(a; c; z) = \sum_{n=0}^{\infty}\frac{(a)_n\, z^n}{(c)_n\, n!} \tag{8.184}$$

where

$$(a)_0 = 1 \qquad \text{and} \qquad (a)_n = a(a+1)(a+2)\dots(a+n-1) \quad \text{for } n > 0.$$

It is understood that c is not equal to a negative integer. An alternate solution of Kummer's differential equation can also be written in terms of a Kummer function. In particular

$$w_2(z) = z^{1-c}\,{}_1F_1(a-c+1; 2-c; z). \tag{8.185}$$

For our purposes, we can only consider solutions which are behaved well enough near $r = 0$ so that

$$\int_0^{+\infty} [(F(r))^2 + (G(r))^2]\, \mathrm{d}r < +\infty. \tag{8.186}$$

This condition is restrictive enough to eliminate the alternate solutions. The same condition is also quite restrictive on the specific form of the solutions that appear in Eq. (8.184). If Re $z \to +\infty$, then (see Magnus, Oberhettinger and Soni 1966, p. 289):

$${}_1F_1(a; c; z) \sim \frac{\Gamma(c)}{\Gamma(a)}\, \mathrm{e}^z z^{a-c}.$$

This means that for most values of a, Kummer's function blows up too fast near infinity for the integral in Eq. (8.186) to converge. In order for the integral to converge, it is necessary for a to be either 0 or some negative integer. In this circumstance, Kummer's function reduces to a polynomial and the wave function is suitably well behaved at infinity. The requirement that a be a nonpositive integer determines the energy eigenvalues for hydrogen-like atoms. Comparing Eq. (8.181) with Eq. (8.183), we note that

$$a = s - A = -n.$$

Applying Eq. (8.168), we have

$$Z\alpha E = (s+n)(m^2c^4 - E^2)^{\frac{1}{2}}. \tag{8.187}$$

Squaring both sides of Eq. (8.187) and solving for E^2 gives us

$$E^2 = m^2c^4\left[1 + \frac{Z^2\alpha^2}{(s+n)^2}\right]^{-1}$$

From Eq. (8.187), it is clear that we must take the positive square root. Thus

$$E = mc^2\left[1 + \frac{Z^2\alpha^2}{(s+n)^2}\right]^{-\frac{1}{2}} \tag{8.188}$$

where $s = (k^2 - Z^2\alpha^2)^{\frac{1}{2}}$ and $n = 0, 1, 2, \ldots$.

Except for a small correction known as the Lamb shift, these eigenvalues agree very well with experiment.

Now let us return to the computation of the eigenfunctions. Since $A - s = n$, Eqs. (8.181) and (8.117) become

$$\rho\frac{\mathrm{d}^2 f_1}{\mathrm{d}\rho^2} + [(2s+1) - \rho]\frac{\mathrm{d}f_1}{\mathrm{d}\rho} + nf_1 = 0 \tag{8.189}$$

and

$$\rho\frac{\mathrm{d}^2 f_2}{\mathrm{d}\rho^2} + [(2s+1) - \rho]\frac{\mathrm{d}f_2}{\mathrm{d}\rho} + (n-1)f_2 = 0 \tag{8.190}$$

The polynomials which satisfy these slightly specialized versions of Kummer's equation are known as *generalized Laguerre polynomials.* In particular

$$f_1(\rho) = L_n^{(2s)}(\rho) = \rho^{-2s}\frac{\mathrm{e}^{\rho}}{n!}\frac{\mathrm{d}^n}{\mathrm{d}\rho^n}[\mathrm{e}^{-\rho}\rho^{n+2s}] \tag{8.191}$$

for $n = 0, 1, 2, 3, \ldots$, and

$$f_2(\rho) = L_{n-1}^{(2s)}(\rho) = \rho^{-2s}\frac{\mathrm{e}^{\rho}}{(n-1)!}\frac{\mathrm{d}^{n-1}}{\mathrm{d}\rho^{n-1}}[\mathrm{e}^{-\rho}\rho^{n-1+2s}] \tag{8.192}$$

for $n = 1, 2, 3, \ldots$.

To determine $F(r)$ and $G(r)$, we must determine $F_1(r)$ and $F_2(r)$. To do this we must determine the constants c_1 and c_2 that appear in Eqs. (8.174) and (8.175). To do this, we note that sine $A - s = n$,

$$\frac{c_1}{B_+c_2}\rho^{n+1}\frac{\mathrm{d}}{\mathrm{d}\rho}(\rho^{-n}L_n^{(2s)}(\rho)) = L_{n-1}^{(2s)}(\rho). \tag{8.193}$$

From Eq. (8.191), it is clear that the lowest order term in the polynomial $L_n^{(2s)}(\rho)$ is the constant $\Gamma(n+2s+1)/(n!\Gamma(2s+1))$. Furthermore, the lowest order term in the polynomial $L_{n-1}^{(2s)}(\rho)$ is the constant

$$\Gamma(n+2s)/[(n-1)!\Gamma(2s+1)].$$

(Consistent with this constant, I will use the convention that

$$f_2(\rho) = L_{n-1}^{(2s)}(\rho) = 0$$

when $n = 0$.) Using these values to get the lowest order terms on both sides of Eq. (8.193), we have

$$\frac{-c_1}{B_+c_2}\frac{\Gamma(n+2s+1)}{(n-1)!\,\Gamma(2s+1)} = \frac{\Gamma(n+2s)}{(n-1)!\,\Gamma(2s+1)}$$

or

$$\frac{-c_1}{B_+c_2}(n+2s) = 1.$$

From this, it can be shown that

$$c_1 = \mu B_+ = \mu(k + [(s+n)^2 + Z\alpha^2]^{\frac{1}{2}}) = \mu(k + [k^2 + 2sn + n^2]^{\frac{1}{2}}) \tag{8.194}$$

and

$$c_2 = -\mu(n+2s) \tag{8.195}$$

where μ is some appropriate normalizing constant.

To summarize, we have

$$F(r) = \frac{(\alpha_+)^{\frac{1}{2}}}{2\lambda}(F_1(r) + F_2(r)), \tag{8.196}$$

$$G(r) = \frac{(\alpha_-)^{\frac{1}{2}}}{2\lambda}(F_2(r) - F_1(r)), \tag{8.197}$$

$$F_1(r) = c_1\rho^s\,\mathrm{e}^{-\rho/2}L_n^{(2s)}(\rho), \tag{8.198}$$

$$F_2(r) = c_2\rho^s\,\mathrm{e}^{-\rho/2}L_{n-1}^{(2s)}(\rho), \tag{8.199}$$

$$\rho = 2\lambda r = 2(\alpha_+\alpha_-)^{\frac{1}{2}}r, \tag{8.200}$$

$$\alpha_+ = \frac{mc^2 + E}{\hbar c}, \quad \text{and} \quad \alpha_- = \frac{mc^2 - E}{\hbar c}.$$

If $\int_0^\infty (F(r)^2 + G(r)^2\,\mathrm{d}r = 1$, then

$$(c_1)^2 + (c_2)^2 = \frac{4\hbar\lambda^3}{mc}. \tag{8.201}$$

To compare the energy levels with those computed for the nonrelativistic

Schrödinger equation, it is necessary to expand the right-hand side of Eq. (8.188) in powers of the fine structure constant. If one identifies the *total quantum number* n_t of the Schrödinger theory with $|k| + n$, then one gets

$$E \approx mc^2\left[1 - \frac{Z^2\alpha^2}{2(n_t)^2} - \frac{Z^4\alpha^4}{2(n_t)^4}\left(\frac{n_t}{|k|} - \frac{3}{4}\right)\right]. \tag{8.202}$$

The first term on the right side of Eq. (8.202) represents the rest energy of the electron. The second term is

$$-\frac{mc^2Z^2\alpha^2}{2(n_t)^2} = -\frac{mZ^2e^4}{8h^2(n_t)^2}.$$

This term gives us the Schrödinger energy levels. The third term, which must be regarded as a relativistic correction, gives the fine structure splitting of the Schrödinger energy levels.

Now let us consider the problem of determining the product $\psi\psi^\dagger$ so that we can read off the current and dipole densities. From Eq. (8.153)

$$\psi = \left(I\frac{F(r)}{r} + \mathrm{i}\hat{\gamma}^r\frac{G(r)}{r}\right)\mathscr{Y}^{(\pm)}_{j,m}.$$

Since $\hat{\gamma}^r = -\hat{\gamma}_r$,

$$\psi\psi^\dagger = \left(I\frac{F(r)}{r} - \mathrm{i}\hat{\gamma}_r\frac{G(r)}{r}\right)\mathscr{Y}^{(\pm)}_{j,m}\mathscr{Y}^{(\pm)}_{j,m}\left(I\frac{F(r)}{r} + \mathrm{i}\hat{\gamma}_r\frac{G(r)}{r}\right). \tag{8.203}$$

From Eqs. (8.143) and (8.144),

$$\mathscr{Y}^{(\pm)}_{j,m}\mathscr{Y}^{(\pm)}_{j,m} = \frac{1}{4}\left[IN^m_j(\theta) + \frac{\mathrm{i}\hat{\gamma}_{\theta\phi}}{r\sin\theta}C^m_j(\theta) \pm \frac{\mathrm{i}\hat{\gamma}_{\phi r}}{r\sin\theta}S^m_j(\theta)\right](I + \hat{\gamma}_0).$$

The matrix $\hat{\gamma}_{\theta\phi}$ commutes with $\hat{\gamma}_r$, but $\hat{\gamma}_{\phi r}$ does not. However, $\hat{\gamma}_{\phi r}(I + \hat{\gamma}_0) = \hat{\gamma}_{r\phi 0}(I + \hat{\gamma}_0) = \hat{\gamma}_{0\phi r}(I + \hat{\gamma}_0)$, and $\hat{\gamma}_{0r\phi}$ does commute with $\hat{\gamma}_r$. Thus Eq. (8.203) becomes

$$\begin{aligned}\psi\psi^\dagger &= \frac{1}{4}\left[IN^m_j(\theta) + \frac{\mathrm{i}\hat{\gamma}_{\theta\phi}}{r^2\sin\theta}C^m_j(\theta) + \operatorname{sign}(k)\frac{\mathrm{i}\hat{\gamma}_{0\phi r}}{r\sin\theta}S^m_j(\theta)\right]\\ &\quad\times\left(I\frac{F(r)}{r} - \mathrm{i}\hat{\gamma}_r\frac{G(r)}{r}\right)(I + \hat{\gamma}_0)\left(I\frac{F(r)}{r} + \mathrm{i}\hat{\gamma}_r\frac{G(r)}{r}\right)\end{aligned}$$

or

$$\psi\psi^{\dagger} = \left[I \frac{(F^2 - G^2)}{r^2} + \hat{\gamma}_0 \frac{(F^2 + G^2)}{r^2} + \mathrm{i}\hat{\gamma}_{0r} \frac{2FG}{r^2} \right] \cdot$$
$$\times \frac{1}{4} \left[I N_j^m(\theta) + \frac{\mathrm{i}\hat{\gamma}_{\theta\phi}}{r^2 \sin\theta} C_j^m(\theta) + \mathrm{sign}(k) \frac{\mathrm{i}\hat{\gamma}_{0\phi r}}{r \sin\theta} S_j^m(\theta) \right]. \tag{8.204}$$

According the Eq. (8.19), the components of the 4-current are determined by the equation

$$J^k = e(\hat{\gamma}^k \psi\psi^{\dagger})_0.$$

If we normalize the Clifford wave function so that $\int_0^\infty (F^2 + G^2)\,\mathrm{d}r = 1$, then the factor of $\frac{1}{4}$ disappears and we have

$$J^0 = e \frac{(F(r))^2 + (G(r))^2}{r^2} N_j^m(\theta), \tag{8.205}$$

$$J^r = J^\theta = 0, \tag{8.206}$$

and

$$J^\phi = \mathrm{sign}(k) e \frac{2F(r)G(r)}{r^3 \sin\theta} S_j^m(\theta). \tag{8.207}$$

For the dipole moments, we note that from Eq. (8.28) we have

$$M^{jk} = -\frac{ie\hbar}{2mc} (\hat{\gamma}^{jk} \psi\psi^{\dagger})_0.$$

Using the same normalization that we used for the 4-current, we have

$$M^{r0} = \frac{e\hbar}{2mc} \frac{2F(r)G(r)}{r^2} N_j^m(\theta), \tag{8.208}$$

$$M^{\theta\phi} = -\frac{e\hbar}{2mc} \frac{(F(r))^2 - (G(r))^2}{r^4 \sin\theta} C_j^m(\theta), \tag{8.209}$$

$$M^{\phi r} = -\mathrm{sign}(k) \frac{e\hbar}{2mc} \frac{(F(r))^2 + (G(r))^2}{r^3 \sin\theta} S_j^m(\theta), \tag{8.210}$$

and

$$M^{\theta 0} = M^{\phi 0} = M^{r\theta} = 0. \tag{8.211}$$

Problem 8.26. Show that the polynomial defined by Eq. (8.191) can also be written in the form

$$L_n^{(2s)}(\rho) = \sum_{m=0}^{n} (-1)^m \binom{n+2s}{n-m} \frac{\rho^m}{m!} \tag{8.212}$$

where

$$\binom{n+2s}{n-m} = \frac{(n+2s)(n+2s-1)\ldots(m+2s+1)}{(n-m)!}.$$

Problem 8.27. Use Eq. (8.212) to show that the generalized Laguerre polynomial is a solution to Eq. (8.189).

Problem 8.28. (The results of this problem show how the nonrelativistic solutions of Schrödinger's equation for the hydrogen atom can be obtained as limits of the Dirac solutions.) Suppose the normalization constant that appears in Eqs. (8.194) and (8.195) is adjusted so it is consistent with Eq. (8.201). Consider the consequence of then taking the limit $c \to \infty$ and permitting the fine structure constant to go to zero.

(1) Show $\lim_{c\to\infty} G(r) = 0$.
(2) For $k > 0$, show that

$$\lim_{c\to\infty} \frac{F(r)}{r} = \text{const} \times \rho^{k-1}\, e^{-\rho/2}[L_n^{(2k)}(\rho) - L_{n-1}^{(2k)}(\rho)]$$

where $\rho = [2Za_0 r/(k+n)]$ and a_0 is the *Bohr radius*, that is, $a_0 = 4\pi\hbar^2/me^2$.
(3) For $k < 0$, show

$$\lim_{c\to\infty} \frac{F(r)}{r} = \text{const.} \times \rho^{|k|-1}\, e^{-\rho/2}[nL_n^{(2|k|)}(\rho) - (2|k|+n)L_{n-1}^{(2|k|)}(\rho)],$$

where $\rho = (2Za_0 r)/(|k|+n)$.
(4) From the material covered in Section 8.4, it is clear that for $k > 0$, $k = j + \frac{1}{2} = l + 1$. Also according to Magnus, Oberhettinger, and Soni (1966, p. 241), we have

$$L_n^{(q)}(x) = L_n^{(q+1)}(x) - L_{n-1}^{(q+1)}(x).$$

Use this relation to show that for $k > 0$,

$$\lim_{c\to\infty} \frac{F(r)}{r} = \text{const.} \times \rho^{l}\, e^{-\rho/2} L_n^{(2l+1)}(\rho).$$

(5) From Section 8.4, it is clear that for $k < 0$, $|k| = j + \frac{1}{2} = l$. Also from Magnus, Oberhettinger, and Soni (1966, p. 241)

$$xL_n^{(q+1)}(x) = (n + q + 1)L_n^{(q)}(x) - (n + 1)L_{n+1}^{(q)}(x).$$

Use this relation along with the result of part (3) to show that for $k < 0$,

$$\lim_{c \to \infty} \frac{F(r)}{r} = \text{const.} \times \rho^l \, e^{-\rho/2} L_{n-1}^{(2l+1)}(\rho).$$

Note: the results of Problem 8.28 enable one to associate the solutions of Dirac's equation with the solutions of the nonrelativistic Schrödinger equation. From the discussion immediately preceding Eq. (8.202), one sees that the *total quantum number* n_t that appears in the Schrödinger model may be identified with the sum $|k| + n$. Thus for $k > 0$, $n_t = l + 1 + n$ and we may identify n with Schrödinger's radial quantum number n_r since $n_t = n_r + l + 1$. On the other hand, for $k < 0$, $n_t = l + n$ and n must be identified with the sum $n_r + 1$. In either case, we see that

$$\lim_{c \to \infty} \frac{F(r)}{r} = \text{const.} \times \rho^l \, e^{-\rho/2} L_{n_r}^{(2l+1)}(\rho)$$

is the radially dependent factor in Schrödinger's wave function.

Problem 8.29. Show that Eq. (8.204) can be recast in the form

$$\psi\psi^\dagger = \frac{N(F^2 - G^2)}{r^2} P_0 P_{12}$$

where

$$P_0 = \frac{1}{2}\left(I + \hat{\gamma}_0 \frac{F^2 + G^2}{F^2 - G^2} + \frac{\hat{\gamma}_\phi}{r \sin\theta} \frac{2FGS}{(F^2 - G^2)N} - J \frac{2FGC}{(F^2 - G^2)N}\right)$$

and

$$P_{12} = \frac{1}{2}\left[I + \mathrm{i}\frac{(F^2 + G^2)C}{A}\left(\frac{\hat{\gamma}_{\theta\phi}}{r^2 \sin\theta}(F^2 + G^2)N - \frac{\hat{\gamma}_{0\theta}}{r} 2FGS\right)\right.$$
$$\left. + \, \mathrm{i}\frac{(F^2 - G^2)S}{A}\left(\frac{\hat{\gamma}_{\phi r}}{r \sin\theta}(F^2 + G^2)N + \hat{\gamma}_{0r} 2FGS\right)\right]$$

where

$$N = N_j^m(\theta), \qquad S = \operatorname{sign}(k) S_j^m(\theta), \qquad C = C_j^m(\theta),$$

and

$$A = (F^2 + G^2)^2 N^2 - 4F^2 G^2 S^2.$$

Note: P_0 and P_{12} are commuting projection operators. Aside from a normalization factor, P_0 is the result of applying a duality rotation and a boost operator in the ϕ-direction to the projection operator $\frac{1}{2}(I + \hat{\gamma}_0)$. P_{12} is the result of applying a rotation operator and the same boost operator to the projection operator $\frac{1}{2}(I + \mathrm{i}\hat{\gamma}_{12})$.

9

THE KERR METRIC BY AN ELEMENTARY BRUTE FORCE METHOD

9.1 The Kerr Metric

As already observed, Schwarzschild constructed a spherically symmetric solution to Einstein's system of field equations in 1916. However, the exact solution for a cylindrically symmetric rotating body was not discovered until 1963 by Roy P. Kerr. This solution has since become the focus of almost all mathematical investigations into the nature of black holes.

Professor Kerr (1963) used the formalism of spinors to obtain his solution. It would be quite awkward to use the formalism of differential forms to obtain the solution. However, Clifford algebra provides an alternative computational tool which is both powerful and elementary.

Perhaps the most elementary derivation of the Kerr metric that has been devised is that which appears in *Introduction to General Relativity* by Adler, Bazin, and Schiffer (1975). Those authors were able to construct the Kerr solution of Einstein's field equations by a very skillful use of vector algebra. By using Clifford algebra, it is possible to streamline some of their computations. The use of Fock–Ivanenko 2-vectors is particularly helpful in reducing Einstein's field equations to Eq. (9.34) below.

The Kerr metric is a degenerate metric that can be characterized by the equation:

$$ds = \gamma_\alpha \, dx^\alpha \tag{9.1}$$

where

$$\gamma_\alpha = \hat{\gamma}_\alpha - m w_\alpha w^k \hat{\gamma}_k. \tag{9.2}$$

In this chapter, the units are chosen so that $c = 1$. Because of the nature of Eq. (9.2), we will abandon our convention that Greek indices be restricted to coordinate systems of Dirac matrices. However, we will continue to use uncapped gammas for coordinate frames and capped gammas for an orthonormal noncoordinate frame. In particular

$$\hat{\gamma}_j \hat{\gamma}_k + \hat{\gamma}_k \hat{\gamma}_j = 2n_{jk}$$

where

$$n_{00} = 1, \qquad n_{11} = n_{22} = n_{33} = -1, \qquad \text{and} \qquad n_{jk} = 0 \quad \text{if } j \neq k.$$

It should also be understood that m is an arbitrary constant which will eventually be interpreted to be the geometric mass of the source and the w_j's represent the components of a null vector.

It is not difficult to show that

$$\gamma^k = \hat{\gamma}^k + mw^k w_j \hat{\gamma}^j, \tag{9.3}$$

$$g_{jk} = n_{jk} - 2mw_j w_k, \tag{9.4}$$

and

$$g^{jk} = n^{jk} + 2mw^j w^k. \tag{9.5}$$

Since $w^j w_j = 0$, it is clear that

$$w^j \gamma_j = w^j(\hat{\gamma}_j - mw_j w^k \hat{\gamma}_k) = w^j \hat{\gamma}_j. \tag{9.6}$$

For much the same reason

$$w_j \gamma^j = w_j \hat{\gamma}^j. \tag{9.7}$$

The Fock–Ivanenko 2-vectors have a particularly simple form:

$$\mathbf{\Gamma}_\alpha = -\frac{m}{2} \gamma^{\eta\nu} \frac{\partial}{\partial x^\eta} (w_\alpha w_\nu). \tag{9.8}$$

(See Problem 5.23.)

From Eq. (5.71), a formula for the curvature 2-form in a coordinate frame is

$$\tfrac{1}{2}\mathcal{R}_{\alpha\beta} = \nabla_\alpha \mathbf{\Gamma}_\beta - \nabla_\beta \mathbf{\Gamma}_\alpha + \mathbf{\Gamma}_\alpha \mathbf{\Gamma}_\beta - \mathbf{\Gamma}_\beta \mathbf{\Gamma}_\alpha. \tag{9.9}$$

For this computation, it is useful to take advantage of Eq. (5.44). In particular

$$\nabla_\alpha \mathbf{\Gamma}_\beta = \frac{\partial}{\partial x^\alpha} \mathbf{\Gamma}_\beta - \mathbf{\Gamma}_\alpha \mathbf{\Gamma}_\beta + \mathbf{\Gamma}_\beta \mathbf{\Gamma}_\alpha. \tag{9.10}$$

With this result, Eq. (9.9) becomes

$$\tfrac{1}{2}\mathcal{R}_{\alpha\beta} = \frac{\partial}{\partial x^\alpha} \mathbf{\Gamma}_\beta - \nabla_\beta \mathbf{\Gamma}_\alpha. \tag{9.11}$$

From Problem (5.14), we know that Einstein's field equations for a vacuum may be written in the form:

$$\tfrac{1}{2}\gamma^{\beta}\mathscr{R}_{\alpha\beta} = \mathbf{0}. \tag{9.12}$$

This means that we are faced with solving the equation:

$$\gamma^{\beta}\frac{\partial}{\partial x^{\alpha}}\boldsymbol{\Gamma}_{\beta} - \boldsymbol{\nabla}\boldsymbol{\Gamma}_{\alpha} = 0. \tag{9.13}$$

To compute the first term in Eq. (9.13), we must determine the formula for $\boldsymbol{\Gamma}_{\beta}$ in the appropriate noncoordinate frame. We first note that

$$\begin{aligned}
\boldsymbol{\Gamma}_{\beta} &= -\frac{m}{2}\gamma^{\eta\nu}\frac{\partial}{\partial x^{\eta}}(w_{\beta}w_{\nu}) \\
&= -\frac{m}{2}(\hat{\gamma}^{\eta} + mw^{\eta}w_{j}\hat{\gamma}^{j}) \wedge (\hat{\gamma}^{\nu} + mw^{\nu}w_{k}\hat{\gamma}^{k})\frac{\partial}{\partial x^{\eta}}(w_{\beta}w_{\nu}) \\
&= -\frac{m}{2}(\hat{\gamma}^{\eta\nu} + mw^{\eta}w_{j}\hat{\gamma}^{j\nu} + mw^{\nu}w_{k}\hat{\gamma}^{\eta k} \\
&\quad + m^{2}w^{\eta}w^{\nu}w_{j}w_{k}\hat{\gamma}^{jk})\frac{\partial}{\partial x^{\eta}}(w_{\beta}w_{\nu})
\end{aligned} \tag{9.14}$$

Because of the symmetry of the dummy indices $w_{j}w_{k}\hat{\gamma}^{jk} = 0$.
Furthermore

$$\begin{aligned}
w^{\nu}\frac{\partial}{\partial x^{\eta}}w_{\nu} &= w^{\nu}\frac{\partial}{\partial x^{\eta}}(n_{\nu\theta}w^{\theta}) = w^{\nu}n_{\nu\theta}\frac{\partial w^{\theta}}{\partial x^{\eta}} \\
&= w_{\theta}\frac{\partial w^{\theta}}{\partial x^{\eta}} = \frac{\partial}{\partial x^{\eta}}(w_{\theta}w^{\theta}) - w^{\theta}\frac{\partial}{\partial x^{\eta}}w_{\theta} \\
&= 0 - w^{\theta}\frac{\partial}{\partial x^{\eta}}w_{\theta}.
\end{aligned}$$

Therefore

$$w^{\nu}\frac{\partial}{\partial x^{\eta}}w_{\nu} = 0. \tag{9.15}$$

It then follows that

$$mw^{\nu}w_{k}\hat{\gamma}^{\eta k}\frac{\partial}{\partial x^{\eta}}(w_{\beta}w_{\nu}) = mw_{k}\hat{\gamma}^{\eta k}\left(w^{\nu}w_{\nu}\frac{\partial w_{\beta}}{\partial x^{\eta}} + w_{\beta}w^{\nu}\frac{\partial}{\partial x^{\eta}}w_{\nu}\right) = 0.$$

With these results, Eq. (9.14) becomes

$$\Gamma_\beta = -\frac{m}{2}\hat{\gamma}^{\eta\nu}\frac{\partial}{\partial x^\eta}(w_\beta w_\nu) - \frac{m^2}{2}\hat{\gamma}^{j\nu}w_j w^\eta \frac{\partial}{\partial x^\eta}(w_\beta w_\nu) \tag{9.16}$$

If we define

$$v_k = w^\eta \frac{\partial}{\partial x^\eta} w_k \tag{9.17}$$

then it is possible to recast the second term on the right-hand side of Eq. (9.16). That is

$$\Gamma_\beta = -\frac{m}{2}\hat{\gamma}^{\eta\nu}\frac{\partial}{\partial x^\eta}(w_\beta w_\nu) - \frac{m^2}{2}\hat{\gamma}^{j\nu}w_j w_\beta v_\nu - \frac{m^2}{2}\hat{\gamma}^{j\nu}w_j w_\nu v_\beta \tag{9.18}$$

Because of the symmetry of the dummy indices in the last term on the right-hand side of Eq. (9.18), that term is zero. We now have

$$\Gamma_\beta = -\frac{m}{2}\hat{\gamma}^{\eta\nu}\frac{\partial}{\partial x^\eta}(w_\beta w_\nu) - \frac{m^2}{2}\hat{\gamma}^{j\nu}w_j w_\beta v_\nu \tag{9.19}$$

We are now faced with obtaining a simple expression for

$$\gamma^\beta \frac{\partial}{\partial x^\alpha}\Gamma_\beta = (\hat{\gamma}^\beta + m w^\beta w_\theta \hat{\gamma}^\theta)\frac{\partial}{\partial x^\alpha}\Gamma_\beta. \tag{9.20}$$

If we now combine Eq. (9.19) with Eq. (9.20) and use the result .of Eq. (9.15) to eliminate the m^3 term, we have

$$\begin{aligned}\gamma^\beta \frac{\partial}{\partial x^\alpha}\Gamma_\beta = &-\frac{m}{2}\hat{\gamma}^\beta\hat{\gamma}^{\eta\nu}\frac{\partial^2}{\partial x^\alpha\,\partial x^\eta}(w_\beta w_\nu)\\ &-\frac{m^2}{2}\hat{\gamma}^\theta\hat{\gamma}^{\eta\nu}w_\theta w^\beta\frac{\partial^2}{\partial x^\alpha\,\partial x^\eta}(w_\beta w_\nu)\\ &-\frac{m^2}{2}\hat{\gamma}^\beta\hat{\gamma}^{j\nu}\frac{\partial}{\partial x^\alpha}(w_j w_\beta v_\nu).\end{aligned} \tag{9.21}$$

We note that $\hat{\gamma}^\beta\hat{\gamma}^{m\nu} = \hat{\gamma}^{\beta m\nu} + n^{\beta m}\hat{\gamma}^\nu - n^{\beta\nu}\hat{\gamma}^m$.

For this reason both 3-vector and 1-vector terms appear in $\gamma^\beta(\partial/\partial x^\alpha)\Gamma_\beta$. However, because of the symmetry of the dummy indices, it is obvious that

two of these 3-vector terms are zero. The remaining 3-vector term is

$$-\frac{m^2}{2}\,\hat{\gamma}^{\theta\eta\nu} w_\theta w^\beta \frac{\partial^2}{\partial x^\alpha\,\partial x^\eta}(w_\beta w_\nu) = -\frac{m^2}{2}\,\hat{\gamma}^{\theta\eta\nu} w_\theta w_\nu w^\beta \frac{\partial^2}{\partial x^\alpha\,\partial x^\eta} w_\beta$$

+ (terms which are zero either because $w^\beta w_\beta = 0$ or because of Eq. (9.15)

The remaining 3-vector term is then also zero because of the symmetry of the θ and ν indices. Writing down the 1-vector terms now gives us

$$\begin{aligned}
\gamma^\beta \frac{\partial}{\partial x^\alpha}\mathbf{\Gamma}_\beta = &-\frac{m}{2}(n^{\beta\eta}\hat{\gamma}^\nu - n^{\beta\nu}\hat{\gamma}^\eta)\frac{\partial^2}{\partial x^\alpha\,\partial x^\eta}(w_\beta w_\nu) \\
&-\frac{m^2}{2}(n^{\theta\eta}\hat{\gamma}^\nu - n^{\theta\nu}\hat{\gamma}^\eta) w_\theta w^\beta \frac{\partial^2}{\partial x^\alpha\,\partial x^\eta}(w_\beta w_\nu) \\
&-\frac{m^2}{2}(n^{\beta j}\hat{\gamma}^\nu - n^{\beta\nu}\hat{\gamma}^j)\frac{\partial}{\partial x^\alpha}(w_j w_\beta v_\nu).
\end{aligned} \tag{9.22}$$

When we use the n^{jk}'s to raise indices, some terms are eliminated simply because $w^k w_k = 0$. Other terms disappear because, from the definition of v_k in Eq. (9.17), it is clear that $w^\beta v_\beta = 0$. Finally a consequence of Eq. (9.15) is that

$$\hat{\gamma}^\eta w^\nu w^\beta \frac{\partial^2}{\partial x^\alpha\,\partial x^\eta}(w_\beta w_\nu) = 0.$$

Applying these eliminations, Eq. (9.22) becomes

$$\gamma^\beta \frac{\partial}{\partial x^\alpha}\mathbf{\Gamma}_\beta = -\frac{m}{2}\left[\hat{\gamma}^\nu \frac{\partial^2}{\partial x^\alpha\,\partial x^\eta}(w^\eta w_\nu)\right] - \frac{m^2}{2}\left[\hat{\gamma}^\nu w^\eta w^\beta \frac{\partial^2}{\partial x^\alpha\,\partial x^\eta}(w_\beta w_\nu)\right]. \tag{9.23}$$

We now turn to the task of obtaining a reasonably simple expression for $\nabla\mathbf{\Gamma}_\alpha$. We first note that

$$\begin{aligned}
\mathbf{\Gamma}_\beta &= -\frac{m}{2}\gamma^{\eta\nu}\frac{\partial}{\partial x^\eta}(w_\beta w_\nu) \\
&= -\frac{m}{2}\gamma^\eta \wedge \gamma^\nu \nabla_\eta(w_\beta w_\nu) \\
&= -\frac{m}{2}\gamma^\eta \wedge \nabla_\eta(\gamma^\nu w_\beta w_\nu) + \frac{m}{2}\gamma^\eta \wedge \Gamma_{\eta\theta}{}^\nu \gamma^\theta w_\beta w_\nu \\
&= -\frac{m}{2}\mathbf{d}(\gamma^\nu w_\beta w_\nu) + 0.
\end{aligned}$$

Thus it follows that

$$\mathbf{d}\boldsymbol{\Gamma}_\beta = -\frac{m}{2}\,\mathbf{dd}(\gamma^\nu w_\beta w_\nu) = 0. \tag{9.24}$$

In turn, this implies that

$$\boldsymbol{\nabla}\boldsymbol{\Gamma}_\alpha = (\boldsymbol{\delta} + \mathbf{d})\boldsymbol{\Gamma}_\alpha = \boldsymbol{\delta}\boldsymbol{\Gamma}_\alpha \tag{9.25}$$

This means that if we can express $\boldsymbol{\Gamma}_\alpha$ in the form $\boldsymbol{\Gamma}_\alpha = \frac{1}{2}F_\alpha{}^{\theta\phi}\gamma_{\theta\phi}$ then we can use Eq. (7.17) to get

$$\boldsymbol{\delta}\boldsymbol{\Gamma}_\alpha = \frac{1}{\sqrt{|g|}}\frac{\partial}{\partial x^\beta}(\sqrt{|g|}\,F_\alpha{}^{\beta\phi})\gamma_\phi. \tag{9.26}$$

(It should be noted that a review of the derivation of Eq. (7.17) reveals that it remains valid when the coefficients of the p-vectors have some unsummed indices.)

To use Eq. (9.26) we need to know the determinant of $g_{\alpha\beta}$. We note that $g_{\alpha\beta} = n_{\alpha\beta} - 2mw_\alpha w_\beta$. Furthermore by a spatial rotation, we can have $w_1 = w_0$ and $w_2 = w_3 = 0$. Thus

$$g = \det\begin{bmatrix} 1 - 2m(w_0)^2 & -2m(w_0)^2 & 0 & 0 \\ -2m(w_0)^2 & -1 - 2m(w_0)^2 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} = -1.$$

Now

$$\begin{aligned}
\boldsymbol{\Gamma}_\alpha &= -\frac{m}{2}\gamma^{\eta\nu}\frac{\partial}{\partial x^\eta}(w_\alpha w_\nu) \\
&= -\frac{m}{2}\gamma_{\theta\phi}g^{\eta\theta}g^{\nu\phi}\frac{\partial}{\partial x^\eta}(w_\alpha w_\nu) \\
&= -\frac{m}{2}\gamma_{\theta\phi}g^{\eta\theta}(n^{\nu\phi} + 2mw^\nu w^\phi)\frac{\partial}{\partial x^\eta}(w_\alpha w_\nu) \\
&= -\frac{m}{2}\gamma_{\theta\phi}g^{\eta\theta}\frac{\partial}{\partial x^\eta}(w_\alpha w^\phi) \\
&= -\frac{m}{2}\gamma_{\theta\phi}(n^{\eta\theta} + 2mw^\eta w^\theta)\frac{\partial}{\partial x^\eta}(w_\alpha w^\phi)
\end{aligned}$$

or

$$\mathbf{\Gamma}_\alpha = -\frac{m}{2}\gamma_{\theta\phi}n^{\eta\theta}\frac{\partial}{\partial x^\eta}(w_\alpha w^\phi) - m^2\gamma_{\theta\phi}w^\theta[v_\alpha w^\phi + w_\alpha v^\phi].$$

Since $\gamma_{\theta\phi}w^\theta v_\alpha w^\phi = 0$ and $\gamma_{\theta\phi}A^{\theta\phi} = \frac{1}{2}\gamma_{\theta\phi}(A^{\theta\phi} - A^{\phi\theta})$, we have

$$\begin{aligned}\mathbf{\Gamma}_\alpha = &-\frac{m}{4}\gamma_{\theta\phi}\left[n^{\eta\theta}\frac{\partial}{\partial x^\eta}(w_\alpha w^\phi) - n^{\eta\phi}\frac{\partial}{\partial x^\eta}(w_\alpha w^\theta)\right]\\ &-\frac{m^2}{2}\gamma_{\theta\phi}[w^\theta v^\phi w_\alpha - w^\phi v^\theta w_\alpha].\end{aligned}$$

We can now use Eq. (9.26) to get

$$\begin{aligned}\boldsymbol{\delta}\mathbf{\Gamma}_\alpha = &-\frac{m}{2}\gamma_\phi\left[n^{\eta\beta}\frac{\partial^2}{\partial x^\beta\,\partial x^\eta}(w_\alpha w^\phi) - n^{\eta\phi}\frac{\partial^2}{\partial x^\beta\,\partial x^\eta}(w_\alpha w^\beta)\right]\\ &- m^2\gamma_\phi\left[\frac{\partial}{\partial x^\beta}(w^\beta w_\alpha v^\phi) - \frac{\partial}{\partial x^\beta}(w^\phi w_\alpha v^\beta)\right]\\ = &-\frac{m}{2}(\hat{\gamma}_\phi - m w_\phi w^k\hat{\gamma}_k)\left[\Box^2(w_\alpha w^\phi) - n^{\eta\phi}\frac{\partial^2}{\partial x^\beta\,\partial x^\eta}(w_\alpha w^\beta)\right]\\ &- m^2(\hat{\gamma}_\phi - m w_\phi w^k\hat{\gamma}_k)\left[\frac{\partial}{\partial x^\beta}(w^\beta w_\alpha v^\phi) - \frac{\partial}{\partial x^\beta}(w^\phi w_\alpha v^\beta)\right].\end{aligned} \tag{9.27}$$

We note that $\hat{\gamma}_\phi\Box^2(w_\alpha w^\phi) = \hat{\gamma}_\phi\Box^2(w_\alpha n^{\nu\phi}w_\nu) = \hat{\gamma}^\nu\Box^2(w_\alpha w_\nu)$. Using this and similar manipulations, along with a slightly modified version of Eq. (9.15), Eq. (9.27) becomes

$$\begin{aligned}\boldsymbol{\delta}\mathbf{\Gamma}_\alpha = &-\frac{m}{2}\hat{\gamma}^\nu\left[\Box^2(w_\alpha w_\nu) - \frac{\partial^2}{\partial x^\nu\,\partial x^\beta}(w_\alpha w^\beta)\right]\\ &+\frac{m^2}{2}\hat{\gamma}^p w_p\left[w^\nu\Box^2(w_\alpha w_\nu) - w^\nu\frac{\partial^2}{\partial x^\nu\,\partial x^\beta}(w_\alpha w^\beta)\right]\\ &- m^2\hat{\gamma}^\nu\left[\frac{\partial}{\partial x^\beta}(w^\beta w_\alpha v_\nu - w_\nu w_\alpha v^\beta)\right]\\ &+ m^3\hat{\gamma}^p w_p\left[w^\nu\frac{\partial}{\partial x^\beta}(w^\beta w_\alpha v_\nu)\right].\end{aligned} \tag{9.28}$$

Now we wish to combine Eqs. (9.23) and (9.28) to get Einstein's field equations:

$$\gamma^\beta\frac{\partial}{\partial x^\alpha}\mathbf{\Gamma}_\beta - \boldsymbol{\delta}\mathbf{\Gamma}_\alpha = 0. \tag{9.29}$$

Since m is an arbitrary constant, we will set the coefficients of m, m^2, and m^3 separately equal to zero. However, there is no m^3 term in $\gamma^\beta\, \partial\boldsymbol{\Gamma}_\beta/\partial x^\alpha$. Thus the m^3 coefficient in $\boldsymbol{\delta\Gamma}_\alpha$ must be zero. That is

$$\begin{aligned}
0 &= w^\nu \frac{\partial}{\partial x^\beta}(w^\beta w_\alpha v_\nu)\\
&= w^\nu v_\nu \frac{\partial}{\partial x^\beta}(w^\beta w_\alpha) + w^\nu w^\beta w_\alpha \frac{\partial}{\partial x^\beta} v_\nu\\
&= 0 + w^\beta w_\alpha w^\nu \frac{\partial}{\partial x^\beta} v_\nu\\
&= w^\beta w_\alpha \frac{\partial}{\partial x^\beta}(w^\nu v_\nu) - w^\beta w_\alpha v_\nu \frac{\partial}{\partial x^\beta}(w^\nu)\\
&= 0 - w_\alpha(v_\nu v^\nu).
\end{aligned}$$

Thus $\boldsymbol{v}$ is a null vector which is orthogonal to the null vector $\boldsymbol{w}$. Generally two null vectors which are orthogonal to one another must be scalar multiples of one another. To show this we note that if $v_0 = -Aw_0$, then

$$\boldsymbol{v} = -A(w_0, w_1, w_2, w_3) + (0, u_1, u_2, u_3) = -A\boldsymbol{w} + \boldsymbol{u}. \tag{9.30}$$

Since $0 = \langle \boldsymbol{v}, \boldsymbol{w}\rangle = -A\langle \boldsymbol{w}, \boldsymbol{w}\rangle + \langle \boldsymbol{u}, \boldsymbol{w}\rangle$, it follows that $\langle \boldsymbol{u}, \boldsymbol{w}\rangle = 0$. Also

$$\langle \boldsymbol{v}, \boldsymbol{v}\rangle = A^2\langle \boldsymbol{w}, \boldsymbol{w}\rangle - 2A\langle \boldsymbol{u}, \boldsymbol{w}\rangle + \langle \boldsymbol{u}, \boldsymbol{u}\rangle = 0.$$

This in turn implies that

$$0 = \langle \boldsymbol{u}, \boldsymbol{u}\rangle = n^{ij}u_i u_j = -[(u_1)^2 + (u_2)^2 + (u_3)^2],$$

or $\boldsymbol{u} = \boldsymbol{0}$. Thus from Eq. (9.30), we now have

$$w^\beta \frac{\partial}{\partial x^\beta} w_\theta = +v_\theta = -Aw_\theta. \tag{9.31}$$

Now we introduce one more definition, namely

$$\frac{\partial}{\partial x^\beta} w^\beta = -L. \tag{9.32}$$

Using Eqs. (9.31) and (9.32), we can simplify Eqs. (9.23) and (9.28). We

then have

$$\gamma^{\beta}\frac{\partial}{\partial x^{\alpha}}\mathbf{\Gamma}_{\beta} = \frac{m}{2}\hat{\gamma}^{\nu}\left[\frac{\partial}{\partial x^{\alpha}}((L+A)w_{\nu})\right]$$

$$-\frac{m^2}{2}\hat{\gamma}^{\nu}\left[w^{\eta}\frac{\partial}{\partial x^{\alpha}}\left(w^{\beta}\frac{\partial}{\partial x^{\eta}}(w_{\beta}w_{\nu})\right) - w^{\eta}\frac{\partial w^{\beta}}{\partial x^{\alpha}}\frac{\partial}{\partial x^{\eta}}(w_{\beta}w_{\nu})\right]$$

$$= \frac{m}{2}\hat{\gamma}^{\nu}\left[\frac{\partial}{\partial x^{\alpha}}((L+A)w_{\nu})\right]$$

$$+\frac{m^2}{2}\hat{\gamma}^{\nu}\left[0 + \frac{\partial w^{\beta}}{\partial x^{\alpha}}(-Aw_{\beta}w_{\nu} - Aw_{\beta}w_{\nu})\right],$$

and thus

$$\gamma^{\beta}\frac{\partial}{\partial x^{\alpha}}\mathbf{\Gamma}_{\beta} = \frac{m}{2}\hat{\gamma}^{\nu}\left[\frac{\partial}{\partial x^{\alpha}}((L+A)w_{\nu})\right]. \tag{9.33}$$

Applying the same procedures to Eq. (9.28), we get

$$\boldsymbol{\delta}\mathbf{\Gamma}_{\alpha} = -\frac{m}{2}\hat{\gamma}^{\nu}\left[\Box^2(w_{\alpha}w_{\nu}) + \frac{\partial}{\partial x^{\nu}}((L+A)w_{\alpha})\right]$$

$$+\frac{m^2}{2}\hat{\gamma}^{p}w_{p}\left[w^{\nu}\Box^2(w_{\alpha}w_{\nu}) + w^{\nu}\frac{\partial}{\partial x^{\nu}}((L+A)w_{\alpha})\right].$$

Combining this result with Eq. (9.33), Einstein's field equations become

$$\gamma^{\beta}\frac{\partial}{\partial x^{\alpha}}\mathbf{\Gamma}_{\beta} - \boldsymbol{\delta}\mathbf{\Gamma}_{\alpha} = 0$$

$$= \frac{m}{2}\hat{\gamma}^{\nu}\left[\Box^2(w_{\alpha}w_{\nu}) + \frac{\partial}{\partial x^{\nu}}((L+A)w_{\alpha}) + \frac{\partial}{\partial x^{\alpha}}((L+A)w_{\nu})\right]$$

$$-\frac{m^2}{2}\hat{\gamma}^{p}w_{p}\left[w^{\nu}\Box^2(w_{\alpha}w_{\nu}) + w^{\nu}\frac{\partial}{\partial x^{\nu}}((L+A)w_{\alpha})\right].$$

Since the coefficient of $m = 0$, we finally have

$$\Box^2(w_{\alpha}w_{\nu}) + \frac{\partial}{\partial x^{\nu}}((L+A)w_{\alpha}) + \frac{\partial}{\partial x^{\alpha}}((L+A)w_{\nu}) = 0. \tag{9.34}$$

Since

$$w^{\nu}\frac{\partial}{\partial x^{\alpha}}((L+A)w_{\nu}) = 0,$$

it follows that the coefficient of m^2 is zero whenever the coefficient of m is zero.

If we now require that the metric tensor be time independent, the equations become substantially more tractable. To pursue this, it is useful to introduce a three component space vector $\boldsymbol{\lambda}$ by the equation:

$$(w_0, w_1, w_2, w_3) = w_0(1, \lambda_1, \lambda_2, \lambda_3). \tag{9.35a}$$

In the immediately ensuing calculations, Latin indices will be restricted to 1, 2, and 3. With this convention

$$\langle \boldsymbol{\lambda}, \boldsymbol{\lambda} \rangle = n^{ij}\lambda_i\lambda_j = -[(\lambda_1)^2 + (\lambda_2)^2 + (\lambda_3)^2] = -1. \tag{9.35b}$$

More importantly, Eq. (9.34) becomes

$$\Box^2(w_0)^2 = n^{ij}\frac{\partial^2}{\partial x^i\,\partial x^j}(w_0)^2 = \nabla^2(w_0)^2 = 0. \tag{9.36a}$$

$$\nabla^2((w_0)^2\lambda_k) = -\frac{\partial}{\partial x^k}((L + A)w_0), \tag{9.36b}$$

and

$$\nabla^2((w_0)^2\lambda_m\lambda_n) = -\frac{\partial}{\partial x^m}((L + A)w_0\lambda_n) - \frac{\partial}{\partial x^n}((L + A)w_0\lambda_m). \tag{9.36c}$$

It should be noted that because of the metric that we are using,

$$\nabla^2 = \left(\gamma^k\frac{\partial}{\partial x^k}\right)^2 = -\left(\frac{\partial^2}{(\partial x^1)^2}, \frac{\partial^2}{(dx^2)^2}, \frac{\partial^2}{(\partial x^3)^2}\right).$$

Expanding the right-hand side of Eq. (9.36c), we have

$$\begin{aligned}\nabla^2((w_0)^2\lambda_m\lambda_n) = &-\frac{\partial\lambda_n}{\partial x^m}(L + A)w_0 - \lambda_n\frac{\partial}{\partial x^m}((L + A)w_0)\\ &-\frac{\partial\lambda_m}{\partial x^n}(L + A)w_0 - \lambda_m\frac{\partial}{\partial x^n}((L + A)w_0).\end{aligned} \tag{9.37}$$

Using Eq. (9.36b) to eliminate the second and fourth terms on the right-hand side of Eq. (9.37), we obtain

$$\begin{aligned}\left(\frac{\partial\lambda_n}{\partial x^m} + \frac{\partial\lambda_m}{\partial x^n}\right)(L + A)w_0 = &-\nabla^2[(w_0)^2\lambda_m\lambda_n]\\ &+ \lambda_n\nabla^2((w_0)^2\lambda_m) + \lambda_m\nabla^2((w_0)^2\lambda_n).\end{aligned} \tag{9.38}$$

Expanding each of the terms on the right-hand side of Eq. (9.38), we get several cancellations. Several of the remaining terms disappear because of Eq. (9.36a). The resulting equation is

$$\frac{\partial \lambda_n}{\partial x^m} + \frac{\partial \lambda_m}{\partial x^n} = \frac{-2w_0}{L + A} n^{ij} \frac{\partial \lambda_m}{\partial x^i} \frac{\partial \lambda_n}{\partial x^j} = -\frac{1}{p} n^{ij} \frac{\partial \lambda_m}{\partial x^i} \frac{\partial \lambda_n}{\partial x^j}, \tag{9.39}$$

where

$$p = \frac{L + A}{2w_0}. \tag{9.40}$$

Equation (9.39) relates the derivatives $\partial \lambda_n / \partial x^m$ to an orthogonal transformation. If we define

$$\bar{\gamma}_k = \hat{\gamma}_k + \frac{1}{p} \hat{\gamma}^m \frac{\partial \lambda_k}{\partial x^m}, \tag{9.41}$$

we note that

$$\begin{aligned}
\langle \bar{\gamma}_k, \bar{\gamma}_j \rangle &= \left\langle \hat{\gamma}_k + \frac{1}{p} \hat{\gamma}^m \frac{\partial \lambda_k}{\partial x^m}, \hat{\gamma}_j + \frac{1}{p} \hat{\gamma}^n \frac{\partial \lambda_j}{\partial x^n} \right\rangle \\
&= \langle \hat{\gamma}_k, \hat{\gamma}_j \rangle + \delta_j^m \frac{1}{p} \frac{\partial \lambda_k}{\partial x^m} + \delta_k^n \frac{1}{p} \frac{\partial \lambda_j}{\partial x^n} + \frac{1}{p^2} n^{mn} \frac{\partial \lambda_k}{\partial x^m} \frac{\partial \lambda_j}{\partial x^n} \\
&= \langle \hat{\gamma}_k, \hat{\gamma}_j \rangle.
\end{aligned} \tag{9.42}$$

From this result, we see that the Clifford numbers $(\bar{\gamma}_1, \bar{\gamma}_2, \bar{\gamma}_3)$ can be obtained from the Clifford numbers $(\hat{\gamma}_1, \hat{\gamma}_2, \hat{\gamma}_3)$ by an orthogonal transformation—that is by a reflection or a rotation. It can be shown that the vector $\boldsymbol{\lambda} = \lambda^j \hat{\gamma}_j$ is invariant under this transformation.

If we use T to designate this transformation, then $T(\hat{\gamma}_j) = \bar{\gamma}_j$ and

$$T(\boldsymbol{\lambda}) = \lambda^j T(\hat{\gamma}_j) = \lambda^j \bar{\gamma}_j = \lambda^j \hat{\gamma}_j + \frac{1}{p} \hat{\gamma}^m \lambda^j \frac{\partial \lambda_j}{\partial x^m}. \tag{9.43}$$

Repeating the argument used for Eq. (9.15), it can be shown that

$$\lambda^j \frac{\partial \lambda_j}{\partial x^m} = 0. \tag{9.44}$$

Thus Eq. (9.43) becomes $T(\boldsymbol{\lambda}) = \lambda^j \hat{\gamma}_j = \boldsymbol{\lambda}$.

It is now clear that our transformation is either a rotation where $\boldsymbol{\lambda}$ is the axis of rotation or a reflection with respect to some plane containing the vector $\boldsymbol{\lambda}$. Useful results follow if we assume it is a rotation. The rotation

operator can be immediately written in the form

$$\begin{aligned}\mathscr{R}(\xi) &= I\cos\frac{\xi}{2} + (\lambda^1\hat{\gamma}_{23} + \lambda^2\hat{\gamma}_{31} + \lambda^3\hat{\gamma}_{12})\sin\frac{\xi}{2} \\ &= I\cos\frac{\xi}{2} - \hat{J}(\lambda^k\hat{\gamma}_k)\sin\frac{\xi}{2} \\ &= I\cos\frac{\xi}{2} - \hat{J}\boldsymbol{\lambda}\sin\frac{\xi}{2},\end{aligned} \tag{9.45}$$

where $\hat{J} = \hat{\gamma}_{123}$.

Using the fact that $\hat{J}$ commutes with all Dirac matrices and $\hat{J}^2 = I$, we have

$$\begin{aligned}\bar{\gamma}_j &= \hat{\gamma}_j + \frac{1}{p}\hat{\gamma}^i\frac{\partial\lambda_j}{\partial x^i} \\ &= \mathscr{R}(\xi)\hat{\gamma}_j\mathscr{R}^{-1}(\xi) \\ &= \left[I\cos\frac{\xi}{2} - \hat{J}\boldsymbol{\lambda}\sin\frac{\xi}{2}\right]\hat{\gamma}_j\left[I\cos\frac{\xi}{2} + \hat{J}\boldsymbol{\lambda}\sin\frac{\xi}{2}\right] \\ &= \hat{\gamma}_j\cos^2\frac{\xi}{2} + \hat{J}(\hat{\gamma}_j\boldsymbol{\lambda} - \boldsymbol{\lambda}\hat{\gamma}_j)\sin\frac{\xi}{2}\cos\frac{\xi}{2} - \boldsymbol{\lambda}\hat{\gamma}_j\boldsymbol{\lambda}\sin^2\frac{\xi}{2}.\end{aligned} \tag{9.46}$$

To simplify this expression, we observe that

$$\boldsymbol{\lambda}\hat{\gamma}_j\boldsymbol{\lambda} = \lambda^i\lambda^k\hat{\gamma}_i\hat{\gamma}_j\hat{\gamma}_k = \lambda^i\lambda^k(\hat{\gamma}_{ijk} + n_{ij}\hat{\gamma}_k - n_{ik}\hat{\gamma}_j + n_{jk}\hat{\gamma}_i) = 2\lambda_j\boldsymbol{\lambda} + \hat{\gamma}_j. \tag{9.47}$$

Incorporating Eq. (9.47) into Eq. (9.46) and using the fact that $2\sin\xi/2\cos\xi/2 = \sin\xi$ and $2\sin^2\xi/2 = 1 - \cos\xi$, we have

$$\hat{\gamma}_j + \frac{1}{p}\hat{\gamma}^m\frac{\partial\lambda_j}{\partial x^m} = \hat{\gamma}_j\cos\xi - \lambda_j\boldsymbol{\lambda}(1 - \cos\xi) - \hat{J}\boldsymbol{\lambda}\wedge\hat{\gamma}_j\sin\xi$$

or

$$\hat{\gamma}^m\frac{\partial\lambda_j}{\partial x^m} = \mathbf{d}\lambda_j = -\alpha(\hat{\gamma}_j + \lambda_j\boldsymbol{\lambda}) - \hat{J}\beta\boldsymbol{\lambda}\wedge\hat{\gamma}_j, \tag{9.48}$$

where $\alpha = p(1 - \cos\xi)$ and $\beta = p\sin\xi$.

The approach to be used here is to chip away at Eq. (9.48) and eventually obtain an equation for $\alpha + i\beta$. We will then use the solution of that equation to obtain a formula for $\boldsymbol{\lambda}$. For the ensuing calculations, it is useful to note that

$$\hat{\gamma}^i\hat{\gamma}_{ijk} = \hat{\gamma}_{jk}, \tag{9.49}$$

$$\hat{\gamma}^i\hat{\gamma}_{ij} = 2\hat{\gamma}_j, \tag{9.50}$$

and

$$\hat{\gamma}^i \hat{\gamma}_i = 3I. \tag{9.51}$$

Furthermore if $\boldsymbol{a}$ is a 1-vector then

$$\boldsymbol{\lambda}(\boldsymbol{\lambda} \wedge \boldsymbol{a}) = \langle \boldsymbol{\lambda}, \boldsymbol{\lambda} \rangle \boldsymbol{a} - \langle \boldsymbol{\lambda}, \boldsymbol{a} \rangle \boldsymbol{\lambda} = -\boldsymbol{a} - \langle \boldsymbol{\lambda}, \boldsymbol{a} \rangle \boldsymbol{\lambda}, \tag{9.52}$$

$$\boldsymbol{\lambda}(\boldsymbol{\lambda} \wedge \boldsymbol{a}) + (\boldsymbol{\lambda} \wedge \boldsymbol{a})\boldsymbol{\lambda} = 0, \tag{9.53}$$

and

$$\begin{aligned}(\boldsymbol{\lambda} \wedge \boldsymbol{a})(\boldsymbol{\lambda} \wedge \boldsymbol{a}) &= \langle \boldsymbol{\lambda}, \boldsymbol{\lambda} \rangle \langle \boldsymbol{a}, \boldsymbol{a} \rangle + (\langle \boldsymbol{\lambda}, \boldsymbol{a} \rangle)^2 \\ &= \langle \boldsymbol{a}, \boldsymbol{a} \rangle + (\langle \boldsymbol{\lambda}, \boldsymbol{a} \rangle)^2. \end{aligned} \tag{9.54}$$

(See Problem 9.1.)

Multiplying both sides of Eq. (9.48) by $\boldsymbol{\lambda}$, using Eq. (9.52) and then projecting out the 0-vectors, we have

$$\langle \boldsymbol{\lambda}, \hat{\gamma}^m \rangle \frac{\partial \lambda_j}{\partial x^m} = -\alpha(\langle \boldsymbol{\lambda}, \hat{\gamma}_j \rangle - \lambda_j)$$

or

$$\lambda^m \frac{\partial \lambda_j}{\partial x^m} = 0. \tag{9.55}$$

Multiplying Eq. (9.48) by $\hat{\gamma}^j$, we have

$$\begin{aligned}\hat{\gamma}^j \hat{\gamma}^m \frac{\partial \lambda_j}{\partial x^m} &= (n^{jm} - \hat{\gamma}^{mj}) \frac{\partial \lambda_j}{\partial x^m} \\ &= \boldsymbol{\delta\lambda} - \mathbf{d}\boldsymbol{\lambda} \\ &= -\alpha(3 + \boldsymbol{\lambda\lambda}) + \hat{J}\beta\hat{\gamma}^j\hat{\gamma}_{jk}\lambda^k \\ &= -2\alpha + 2\hat{J}\beta\boldsymbol{\lambda}.\end{aligned}$$

Equating 0-vectors and then 2-vectors, we have

$$\boldsymbol{\delta\lambda} = \frac{\partial \lambda^m}{\partial x^m} = -2\alpha, \tag{9.56}$$

and

$$\mathbf{d}\boldsymbol{\lambda} = \hat{\gamma}^{mj} \frac{\partial \lambda_j}{\partial x^m} = -2\hat{J}\beta\boldsymbol{\lambda}, \tag{9.57}$$

$$-2\mathbf{d}\alpha = \mathbf{d}\boldsymbol{\delta\lambda} = \hat{\gamma}^m \frac{\partial}{\partial x^m} \frac{\partial \lambda^j}{\partial x^j} = \frac{\partial}{\partial x^j} \hat{\gamma}^m \frac{\partial \lambda^j}{\partial x^m}. \tag{9.58}$$

From Eq. (9.48)

$$\hat{\gamma}^m \frac{\partial \lambda^j}{\partial x^m} = -\alpha(\hat{\gamma}^j + \lambda^j \boldsymbol{\lambda}) + \hat{J}\beta\hat{\gamma}^{jk}\lambda_k$$

and thus

$$-2\mathbf{d}\alpha = -\frac{\partial \alpha}{\partial x^j}(\hat{\gamma}^j + \lambda^j \boldsymbol{\lambda}) - \alpha\left(\frac{\partial \lambda^j}{\partial x^j}\boldsymbol{\lambda} + \lambda^j \frac{\partial \boldsymbol{\lambda}}{\partial x^j}\right)$$
$$+ \hat{J}\hat{\gamma}^{jk}\frac{\partial \beta}{\partial x^j}\lambda_k + \hat{J}\beta\hat{\gamma}^{jk}\frac{\partial \lambda_k}{\partial x^j}. \tag{9.59}$$

From Eq. (9.55), it follows that

$$\lambda^j \frac{\partial \boldsymbol{\lambda}}{\partial x^j} = 0.$$

Equation (9.59) thus becomes

$$-2\mathbf{d}\alpha = -\mathbf{d}\alpha - \langle \boldsymbol{\lambda}, \mathbf{d}\alpha \rangle \boldsymbol{\lambda} - \alpha(\boldsymbol{\delta\lambda})\boldsymbol{\lambda} + \hat{J}(\mathbf{d}\beta \wedge \boldsymbol{\lambda}) + \hat{J}\beta\mathbf{d}\boldsymbol{\lambda}.$$

Using Eqs. (9.56) and (9.57) and then reorganizing the resulting terms gives us

$$\mathbf{d}\alpha = -2(\alpha^2 - \beta^2)\boldsymbol{\lambda} + \langle \boldsymbol{\lambda}, \mathbf{d}\alpha \rangle \boldsymbol{\lambda} + \hat{J}(\boldsymbol{\lambda} \wedge \mathbf{d}\beta). \tag{9.60}$$

By multiplying Eq. (9.60) by $\boldsymbol{\lambda}$ and using Eq. (9.52), we get

$$\boldsymbol{\lambda}\mathbf{d}\alpha = 2(\alpha^2 - \beta^2) - \langle \boldsymbol{\lambda}, \mathbf{d}\alpha \rangle - \hat{J}\mathbf{d}\beta - \hat{J}\langle \boldsymbol{\lambda}, \mathbf{d}\beta \rangle \boldsymbol{\lambda}.$$

Projecting out 0-vectors and then 2-vectors, we discover

$$\langle \boldsymbol{\lambda}, \mathbf{d}\alpha \rangle = \alpha^2 - \beta^2, \tag{9.61}$$

and

$$\boldsymbol{\lambda} \wedge \mathbf{d}\alpha = -\hat{J}\mathbf{d}\beta - \hat{J}\langle \boldsymbol{\lambda}, \mathbf{d}\beta \rangle \boldsymbol{\lambda}. \tag{9.62}$$

Equation (9.61) can be used to revise Eq. (9.60) and we can multiply Eq. (9.62) by $\hat{J}$ to obtain an equation for $\mathbf{d}\beta$. We then have

$$\mathbf{d}\alpha = -(\alpha^2 - \beta^2)\boldsymbol{\lambda} + \hat{J}(\boldsymbol{\lambda} \wedge \mathbf{d}\beta), \tag{9.63}$$

and

$$\mathbf{d}\beta = -\langle \boldsymbol{\lambda}, \mathbf{d}\beta \rangle \boldsymbol{\lambda} - \hat{J}(\boldsymbol{\lambda} \wedge \mathbf{d}\alpha). \tag{9.64}$$

To obtain an expression for $\langle \boldsymbol{\lambda}, \mathbf{d}\beta \rangle$, we go back to Eq. (9.57) and then get

$$\nabla \mathbf{d}\boldsymbol{\lambda} = -2\hat{J}(\nabla\beta)\boldsymbol{\lambda} - 2\hat{J}\beta\nabla\boldsymbol{\lambda}.$$

Projecting out the 3-vectors, one gets

$$0 = -2\hat{J}\langle \mathbf{d}\beta, \boldsymbol{\lambda} \rangle - 2\hat{J}\beta\delta\boldsymbol{\lambda}$$

or

$$\langle \boldsymbol{\lambda}, \mathbf{d}\beta \rangle = -\beta\delta\boldsymbol{\lambda} = 2\alpha\beta. \tag{9.65}$$

If we now replace $\langle \boldsymbol{\lambda}, \mathbf{d}\beta \rangle$ by $2\alpha\beta$ in Eq. (9.64) and then combine the resulting equation with Eq. (9.63), we have

$$\mathbf{d}(\alpha + \mathrm{i}\beta) = -(\alpha + \mathrm{i}\beta)^2\boldsymbol{\lambda} - \mathrm{i}\hat{J}(\boldsymbol{\lambda} \wedge \mathbf{d}(\alpha + \mathrm{i}\beta)). \tag{9.66}$$

Combining Eqs. (9.61) and (9.65), we also have

$$\langle \boldsymbol{\lambda}, \mathbf{d}(\alpha + \mathrm{i}\beta) \rangle = (\alpha + \mathrm{i}\beta)^2. \tag{9.67}$$

From Eq. (9.66), we have

$$\begin{aligned}\nabla \mathbf{d}(\alpha + \mathrm{i}\beta) = &-2(\alpha + \mathrm{i}\beta)(\nabla(\alpha + \mathrm{i}\beta))\boldsymbol{\lambda} - (\alpha + \mathrm{i}\beta)^2\nabla\boldsymbol{\lambda} \\ &- \mathrm{i}\hat{J}\hat{\gamma}^i \frac{\partial}{\partial x^i}\left[\hat{\gamma}^{jk}\lambda_j \frac{\partial}{\partial x^k}(\alpha + \mathrm{i}\beta)\right].\end{aligned}$$

Projecting out the 0-vectors, we have

$$\begin{aligned}\delta\mathbf{d}(\alpha + \mathrm{i}\beta) = &-2(\alpha + \mathrm{i}\beta)\langle \boldsymbol{\lambda}, \mathbf{d}(\alpha + \mathrm{i}\beta) \rangle - (\alpha + \mathrm{i}\beta)^2\delta\boldsymbol{\lambda} \\ &- \mathrm{i}\hat{J}\hat{\gamma}^{ijk}\left[\frac{\partial\lambda_j}{\partial x^i}\frac{\partial}{\partial x^k}(\alpha + \mathrm{i}\beta) + \lambda_j \frac{\partial^2}{\partial x^1\,\partial x^k}(\alpha + \mathrm{i}\beta)\right].\end{aligned} \tag{9.68}$$

Because of the symmetries of the indices,

$$\hat{\gamma}^{ijk}\frac{\partial^2}{\partial x^i\,\partial x^k}(\alpha + \mathrm{i}\beta) = 0.$$

Furthermore

$$\hat{\gamma}^{ijk} = \tfrac{1}{2}(\hat{\gamma}^{ij}\hat{\gamma}^k + \hat{\gamma}^k\hat{\gamma}^{ij}).$$

Using these observations, along with the fact that

$$\langle \boldsymbol{\lambda}, \mathbf{d}(\alpha + \mathrm{i}\beta) \rangle = (\alpha + \mathrm{i}\beta)^2 \qquad \text{and} \qquad \delta\boldsymbol{\lambda} = -2\alpha,$$

Eq. (9.68) becomes

$$\begin{aligned}\boldsymbol{\delta}\mathbf{d}(\alpha + \mathrm{i}\beta) = &-2(\alpha + \mathrm{i}\beta)^3 + 2(\alpha + \mathrm{i}\beta)^2\alpha \\ &- \frac{\mathrm{i}}{2}\hat{J}[(\mathbf{d}\boldsymbol{\lambda})\,\mathrm{d}(\alpha + \mathrm{i}\beta) + (\mathbf{d}(\alpha + \mathrm{i}\beta))\,\mathrm{d}\boldsymbol{\lambda}].\end{aligned}$$

Since $\mathbf{d}\boldsymbol{\lambda} = -2\hat{J}\beta\boldsymbol{\lambda}$, this becomes

$$\begin{aligned}\boldsymbol{\delta}\mathbf{d}(\alpha + \mathrm{i}\beta) = &-2(\alpha + \mathrm{i}\beta)^3 + 2(\alpha + \mathrm{i}\beta)^2\alpha \\ &+ \mathrm{i}\beta(\boldsymbol{\lambda}\mathbf{d}(\alpha + \mathrm{i}\beta) + (\mathbf{d}(\alpha + \mathrm{i}\beta))\boldsymbol{\lambda}).\end{aligned} \tag{9.69}$$

But

$$\begin{aligned}[\boldsymbol{\lambda}\mathbf{d}(\alpha + \mathrm{i}\beta) + (\mathbf{d}(\alpha + \mathrm{i}\beta))\boldsymbol{\lambda}] &= 2\langle\boldsymbol{\lambda}, \mathbf{d}(\alpha + \mathrm{i}\beta)\rangle \\ &= 2(\alpha + i\beta)^2.\end{aligned}$$

Thus Eq. (9.69) finally becomes

$$\boldsymbol{\delta}\mathbf{d}(\alpha + \mathrm{i}\beta) = \boldsymbol{\nabla}^2(\alpha + \mathrm{i}\beta) = 0. \tag{9.70}$$

Equation (9.70) is Laplace's equation which has a wide variety of solutions. However, if we square both sides of Eq. (9.66), we get

$$\begin{aligned}(\mathbf{d}(\alpha + \mathrm{i}\beta))^2 = &(\alpha + \mathrm{i}\beta)^4(\boldsymbol{\lambda})^2 \\ &+ \mathrm{i}\hat{J}[\boldsymbol{\lambda}(\boldsymbol{\lambda} \wedge \mathbf{d}(\alpha + \mathrm{i}\beta)) + (\boldsymbol{\lambda} \wedge \mathbf{d}(\alpha + \mathrm{i}\beta)\boldsymbol{\lambda}](\alpha + \mathrm{i}\beta)^2 \\ &- \hat{J}^2(\boldsymbol{\lambda} \wedge \mathbf{d}(\alpha + \mathrm{i}\beta))^2.\end{aligned}$$

Using Eqs. (9.53) and (9.54), this becomes

$$(\mathbf{d}(\alpha + \mathrm{i}\beta))^2 = -(\alpha + \mathrm{i}\beta)^4 - (\mathbf{d}(\alpha + \mathrm{i}\beta))^2 - (\langle\boldsymbol{\lambda}, \mathbf{d}(\alpha + \mathrm{i}\beta)\rangle)^2.$$

Using Eq. (9.67) one more time and reorganizing the resulting terms in this last equation, we get

$$(\mathbf{d}(\alpha + \mathrm{i}\beta))^2 = -(\alpha + \mathrm{i}\beta)^4. \tag{9.71}$$

Alternatively, this can be written as

$$\left[\mathbf{d}\left(\frac{1}{\alpha + \mathrm{i}\beta}\right)\right]^2 = -1. \tag{9.72}$$

In the context of classical optics, this last equation is known as the eiconal (or eikonal) or characteristic equation. It severely restricts the solutions for Laplace's equation that can be used for our purposes.

Once α and β are determined, it is possible to determine $\boldsymbol{\lambda}$. To pursue this, it is useful to define

$$\rho + \mathrm{i}\sigma = \frac{1}{\alpha + i\beta}. \tag{9.73}$$

Dividing Eq. (9.67) by $(\alpha + \mathrm{i}\beta)^2$, we get

$$-\langle \boldsymbol{\lambda}, \mathbf{d}(\alpha + \mathrm{i}\beta)^{-1} \rangle = 1 \tag{9.74}$$

or

$$\langle \boldsymbol{\lambda}, \mathbf{d}(\rho + \mathrm{i}\sigma) \rangle = -1. \tag{9.75}$$

In much the same fashion, we can divide Eq. (9.66) by $(\alpha + \mathrm{i}\beta)^2$ and obtain

$$\mathbf{d}(\rho + \mathrm{i}\sigma) = \boldsymbol{\lambda} - \mathrm{i}\hat{J}(\boldsymbol{\lambda} \wedge \mathbf{d}(\rho + \mathrm{i}\sigma)).$$

Multiplying this last equation by $\boldsymbol{\lambda}$ and then projecting out the real parts gives us

$$\boldsymbol{\lambda}\mathbf{d}\rho = -I + \hat{J}\boldsymbol{\lambda}(\boldsymbol{\lambda} \wedge \mathbf{d}\sigma).$$

From Eq. (9.52), this becomes

$$\boldsymbol{\lambda}\mathbf{d}\rho = -I - \hat{J}\mathbf{d}\sigma - \hat{J}\langle \boldsymbol{\lambda}, \mathbf{d}\sigma \rangle \boldsymbol{\lambda}.$$

However, from Eq. (9.75), $\langle \boldsymbol{\lambda}, \mathbf{d}\sigma \rangle = 0$. Thus we have $\boldsymbol{\lambda}\mathbf{d}\rho\mathbf{d}\rho = -(I + \hat{J}\mathbf{d}\sigma)\mathbf{d}\rho$ or

$$\boldsymbol{\lambda} = \frac{-(I + \hat{J}\mathbf{d}\sigma)\mathbf{d}\rho}{\langle \mathbf{d}\rho, \mathbf{d}\rho \rangle}. \tag{9.76}$$

It remains for us to determine w_0 in Eqs. (9.36a, b, and c). Since $\alpha = p(1 - \cos \xi)$ and $\beta = p \sin \xi$, we have

$$\alpha^2 + \beta^2 = 2p^2(1 - \cos \xi) = 2p\alpha.$$

From Eq. (9.40)

$$(L + A)w_0 = 2p(w_0)^2 = (w_0)^2(\alpha^2 + \beta^2)/\alpha.$$

Using this relation, Eqs. (9.36a, b, and c) become

$$\nabla^2(w_0)^2 = 0, \tag{9.77a}$$

$$\nabla^2[(w_0)^2\lambda_k] = -\frac{\partial}{\partial x^k}\left[\frac{(w_0)^2}{\alpha}(\alpha^2 + \beta^2)\right], \tag{9.77b}$$

$$\begin{aligned}\nabla^2[(w_0)^2\lambda_m\lambda_n] = &-\frac{\partial}{\partial x^m}\left[\frac{(w_0)^2}{\alpha}(\alpha^2 + \beta^2)\lambda_n\right]\\ &-\frac{\partial}{\partial x^n}\left[\frac{(w_0)^2}{\alpha}(\alpha^2 + \beta^2)\lambda_m\right].\end{aligned} \tag{9.77c}$$

We see that if $(w_0)^2$ is a constant multiple of α, then Eq. (9.77a) is satisfied and Eqs. (9.77b and c) take on a simpler appearance. It can be shown that a constant multiple of α is the most general solution for $(w_0)^2$. We first note that we have solved Eq. (9.39). Reviewing the origin of this equation tells us that Eq. (9.77c) is solved once we have a compatible solution for Eq. (9.77b).

To show that α is a solution for Eq. (9.77b), we need to show that

$$\nabla^2[\alpha\boldsymbol{\lambda}] = -\mathbf{d}(\alpha^2 + \beta^2). \tag{9.78}$$

Using the fact that the Laplacian of α is zero, the left-hand side of Eq. (9.78) can be rewritten as

$$\nabla^2(\alpha\boldsymbol{\lambda}) = 2n^{ki}\frac{\partial\alpha}{\partial x^k}\frac{\partial\boldsymbol{\lambda}}{\partial x^i} + \alpha\nabla^2\boldsymbol{\lambda}. \tag{9.79}$$

To get an expression for $\partial\boldsymbol{\lambda}/\partial x^i$, we note that from Eq. (9.48)

$$\frac{\partial\lambda_j}{\partial x^i} = \left\langle \hat{\gamma}_i, \hat{\gamma}^m\frac{\partial\lambda_j}{\partial x^m}\right\rangle = -\alpha(n_{ij} + \lambda_j\lambda_i) - \langle\hat{\gamma}_i, \hat{J}\beta(\boldsymbol{\lambda}\wedge\hat{\gamma}_j)\rangle. \tag{9.80}$$

To compute $-\langle\hat{\gamma}_1, \hat{J}\beta(\boldsymbol{\lambda}\wedge\hat{\gamma}_j)\rangle$, we project out the 0-vector from the product $-\hat{J}\beta\hat{\gamma}_i(\boldsymbol{\lambda}\wedge\hat{\gamma}_j)$ and thereby get $-\hat{J}\beta\lambda^k\hat{\gamma}_{ikj} = \hat{J}\beta\lambda^k\hat{\gamma}_{jki}$.

To obtain an expression for $\partial\boldsymbol{\lambda}/\partial x^i$, we need to multiply both sides of Eq. (9.80) by $\hat{\gamma}^j$. In so doing, we should observe that an immediate consequence of Eq. (9.49) is that

$$\hat{\gamma}^j(\hat{J}\beta\lambda^k\hat{\gamma}_{jki}) = \hat{J}\beta\lambda^k\hat{\gamma}_{ki} = \hat{J}\beta\boldsymbol{\lambda}\wedge\hat{\gamma}_i.$$

Thus

$$\frac{\partial\boldsymbol{\lambda}}{\partial x^i} = -\alpha(\hat{\gamma}_i + \boldsymbol{\lambda}\lambda_i) + \hat{J}\beta(\boldsymbol{\lambda}\wedge\hat{\gamma}_i), \tag{9.81}$$

and therefore

$$2n^{ki}\frac{\partial\alpha}{\partial x^k}\frac{\partial\boldsymbol{\lambda}}{\partial x^i} = -2\alpha(\mathbf{d}\alpha + \boldsymbol{\lambda}\langle\boldsymbol{\lambda}, \mathbf{d}\alpha\rangle) + 2\hat{J}\beta(\boldsymbol{\lambda} \wedge \mathbf{d}\alpha).$$

Using Eqs. (9.61), (9.64) and (9.65), this becomes

$$2n^{ki}\frac{\partial\alpha}{\partial x^k}\frac{\partial\boldsymbol{\lambda}}{\partial x^i} = -\mathbf{d}(\alpha^2 + \beta^2) - 2\alpha(\alpha^2 + \beta^2)\boldsymbol{\lambda}. \tag{9.82}$$

To get an expression for the second term on the right-hand side of Eq. (9.79), we note that

$$\alpha\boldsymbol{\nabla}^2\boldsymbol{\lambda} = \alpha(\mathbf{d}\boldsymbol{\delta}\boldsymbol{\lambda} + \boldsymbol{\delta}\mathbf{d}\boldsymbol{\lambda}). \tag{9.83}$$

Since $\boldsymbol{\delta}\boldsymbol{\lambda} = -2\alpha$ and $\mathbf{d}\boldsymbol{\lambda} = -2\hat{J}\beta\boldsymbol{\lambda}$, it follows that

$$\alpha\mathbf{d}\boldsymbol{\delta}\boldsymbol{\lambda} = -2\alpha\mathbf{d}\alpha = -\mathbf{d}(\alpha^2) \tag{9.84}$$

and

$$\alpha\boldsymbol{\delta}\mathbf{d}\boldsymbol{\lambda} = \alpha\boldsymbol{\nabla}(-2\hat{J}\beta\boldsymbol{\lambda}) = -2\alpha\hat{J}(\boldsymbol{\nabla}\beta)\boldsymbol{\lambda} - 2\alpha\beta\hat{J}\boldsymbol{\nabla}\boldsymbol{\lambda}.$$

Projecting out the 1-vectors in this last equation gives us

$$\alpha\boldsymbol{\delta}\mathbf{d}\boldsymbol{\lambda} = 2\alpha\hat{J}(\boldsymbol{\lambda} \wedge \mathbf{d}\beta) - 2\alpha\beta\hat{J}\mathbf{d}\boldsymbol{\lambda}$$

Using Eq. (9.63) along with the fact that $\mathbf{d}\boldsymbol{\lambda} = -2\hat{J}\beta\boldsymbol{\lambda}$, this last equation becomes

$$\alpha\boldsymbol{\delta}\mathbf{d}\boldsymbol{\lambda} = \mathbf{d}(\alpha^2) + 2\alpha(\alpha^2 + \beta^2)\boldsymbol{\lambda}. \tag{9.85}$$

Adding Eqs. (9.84) and (9.85), we get

$$\alpha\boldsymbol{\nabla}^2\boldsymbol{\lambda} = 2\alpha(\alpha^2 + \beta^2)\boldsymbol{\lambda}. \tag{9.86}$$

Substituting the results of Eqs. (9.82) and (9.86) into Eq. (9.79), we obtain Eq. (9.78)—the equation we set out to validate.

To show that $(w_0)^2$ must be a constant multiple of α, we substitute $(w_0)^2 = f\alpha$ into Eqs. (9.77a and b) and then show $\mathbf{d}f = 0$.

From Eq. (9.77a)

$$\boldsymbol{\nabla}^2(f\alpha) = (\boldsymbol{\nabla}^2 f)\alpha + 2n^{ij}\frac{\partial f}{\partial x^j}\frac{\partial\alpha}{\partial x^i} + f\boldsymbol{\nabla}^2\alpha = 0$$

or

$$(\boldsymbol{\nabla}^2 f)\alpha + 2n^{ij}\frac{\partial f}{\partial x^j}\frac{\partial\alpha}{\partial x^i} = 0. \tag{9.87}$$

From Eq. (9.77b)

$$\nabla^2(f\alpha\boldsymbol{\lambda}) = -\mathbf{d}(f(\alpha^2 + \beta^2)),$$

or

$$(\nabla^2 f)\alpha\boldsymbol{\lambda} + 2n^{ij}\frac{\partial f}{\partial x^j}\frac{\partial \alpha}{\partial x^i}\boldsymbol{\lambda} + 2n^{ij}\frac{\partial f}{\partial x^j}\alpha\frac{\partial \boldsymbol{\lambda}}{\partial x^i} + f\nabla^2(\alpha\boldsymbol{\lambda})$$
$$= -(\alpha^2 + \beta^2)\mathbf{d}f - f\mathbf{d}(\alpha^2 + \beta^2).$$

Applying Eq. (9.87) and the fact that $\nabla^2(\alpha\boldsymbol{\lambda}) = -\mathbf{d}(\alpha^2 + \beta^2)$, this becomes

$$2\alpha n^{ij}\frac{\partial f}{\partial x^j}\frac{\partial \boldsymbol{\lambda}}{\partial x^i} = -(\alpha^2 + \beta^2)\mathbf{d}f. \tag{9.88}$$

Using Eq. (9.81) for $\partial\boldsymbol{\lambda}/\partial x^i$, we get

$$-2\alpha^2(\mathbf{d}f + \boldsymbol{\lambda}\langle\boldsymbol{\lambda}, \mathbf{d}f\rangle + 2\alpha\beta\hat{J}(\boldsymbol{\lambda} \wedge \mathbf{d}f) = -(\alpha^2 + \beta^2)\mathbf{d}f. \tag{9.89}$$

Multiplying both sides of this equation by $\boldsymbol{\lambda}$, applying Eq. (9.52), and then projecting out first the 0-vectors and then projecting out the 2-vectors, we get

$$0 = -(\alpha^2 + \beta^2)\langle\boldsymbol{\lambda}, \mathbf{d}f\rangle \tag{9.90}$$

and

$$-2\alpha^2(\boldsymbol{\lambda} \wedge \mathbf{d}f) - 2\alpha\beta\hat{J}\mathbf{d}f = -(\alpha^2 + \beta^2)(\boldsymbol{\lambda} \wedge \mathbf{d}f). \tag{9.91}$$

We now see that $\langle\boldsymbol{\lambda}, \mathbf{d}f\rangle = 0$. With this result, Eq. (9.89) becomes

$$(\alpha^2 - \beta^2)\mathbf{d}f = 2\alpha\beta\hat{J}(\boldsymbol{\lambda} \wedge \mathbf{d}f). \tag{9.92}$$

If we multiply Eq. (9.91) by $\hat{J}$ and reorganize the resulting terms, we arrive at a slightly different equation:

$$2\alpha\beta\mathbf{d}f = -(\alpha^2 - \beta^2)\hat{J}(\boldsymbol{\lambda} \wedge \mathbf{d}f). \tag{9.93}$$

If we take the obvious linear combination of Eqs. (9.92) and (9.93), we get

$$[(\alpha^2 - \beta^2)^2 + 4\alpha^2\beta^2]\mathbf{d}f = (\alpha^2 + \beta^2)^2\mathbf{d}f = 0.$$

and thus $\mathbf{d}f = 0$ which is our desired result.

Now let us return to the problem of finding a solution for

$$\nabla^2(\alpha + \mathrm{i}\beta) = 0 \qquad \text{and} \qquad (\mathbf{d}(\rho + \mathrm{i}\sigma))^2 = -1,$$

where

$$\rho + \mathrm{i}\sigma = \frac{1}{\alpha + \mathrm{i}\beta}.$$

An obvious choice is

$$\alpha + \mathrm{i}\beta = 1/r = (x^2 + y^2 + z^2)^{-\frac{1}{2}}.$$

However, this choice leads to what is known as Eddington's form of the Schwarzschild solution. The solution $(x^2 + y^2 + (z - a)^2)^{-\frac{1}{2}}$ would only correspond to a physical translation of the spherically symmetric solution. However, the solution

$$\alpha + \mathrm{i}\beta = (x^2 + y^2 + (z - \mathrm{i}a)^2)^{-\frac{1}{2}} \tag{9.94}$$

does indeed correspond to a physically meaningful solution which is quite distinct from the Schwarzschild solution.

This solution is most naturally expressed in *oblate spheroidal coordinates* (Margenau and Murphy 1956). To understand this coordinate system, consider a family of confocal ellipses and a companion family of confocal hyperbolas in the xz-plane where both families share the same focal points $(\pm a, 0, 0)$. See Figs. 9.1 and 9.2.

In this coordinate system:

$$x = |a| \cosh u \sin \theta \qquad \text{and} \qquad z = |a| \sinh u \cos \theta.$$

The members of the family of ellipses are specified by the value of u. In particular

$$\frac{x^2}{a^2 \cosh^2 u} + \frac{z^2}{a^2 \sinh^2 u} = \sin^2 \theta + \cos^2 \theta = 1. \tag{9.95}$$

On the other hand, members of the family of hyperbolas are determined by the value of θ:

$$\frac{x^2}{a^2 \sin^2 \theta} - \frac{z^2}{a^2 \cos^2 \theta} = \cosh^2 u - \sinh^2 u = 1. \tag{9.96}$$

If ϕ is used to specify the angle between the xz-plane and the plane passing through our point and the z-axis, then:

$$x = |a| \cosh u \sin \theta \cos \phi, \tag{9.97a}$$

$$y = |a| \cosh u \sin \theta \sin \phi, \tag{9.97b}$$

$$z = |a| \sinh u \cos \theta \tag{9.97c}$$

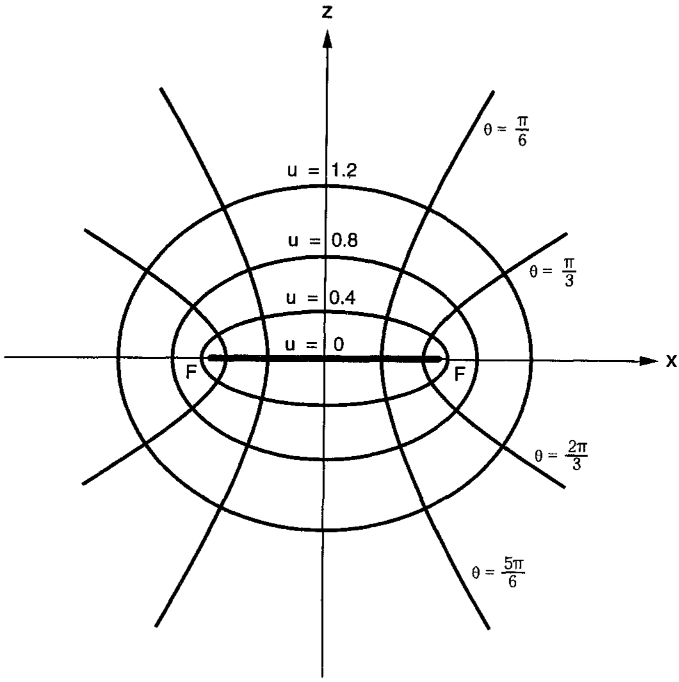


Fig. 9.1. For a fixed value of u, the equation "$x^2/(a^2 \cosh^2 u) + z^2/(a^2 \sinh^2 u) = 1$" defines a member of the family of ellipses with focal points at $x = \pm a$ and $z = 0$. For a fixed value of θ, the equation "$x^2/(a^2 \sin^2 \theta) - z^2/(a^2 \cos^2 \theta) = 1$" defines a member of the family of hyperbolas with the same focal points.

where

$$0 \leqslant u < +\infty, 0 \leqslant \theta \leqslant \pi, \qquad \text{and} \qquad 0 \leqslant \phi < 2\pi.$$

In three dimensions, surfaces of constant u are oblate spheroids, surfaces of constant θ are single sheeted hyperboloids, and surfaces of constant ϕ are planes through the z-axis. (See Fig. 9.3.)

In this coordinate system

$$\begin{aligned} x^2 + y^2 + (z - \mathrm{i}a)^2 &= a^2 \cosh^2 u \sin^2 \theta + (|a| \sinh u \cos \theta - \mathrm{i}a)^2 \\ &= a^2(\sinh^2 u - \cos^2 \theta) - 2\mathrm{i}a|a| \sinh u \cos \theta \\ &= (|a| \sinh u - \mathrm{i}a \cos \theta)^2 \\ &= (\rho + \mathrm{i}\sigma)^2. \end{aligned}$$

Thus

$$\rho = |a| \sinh u \tag{9.98a}$$

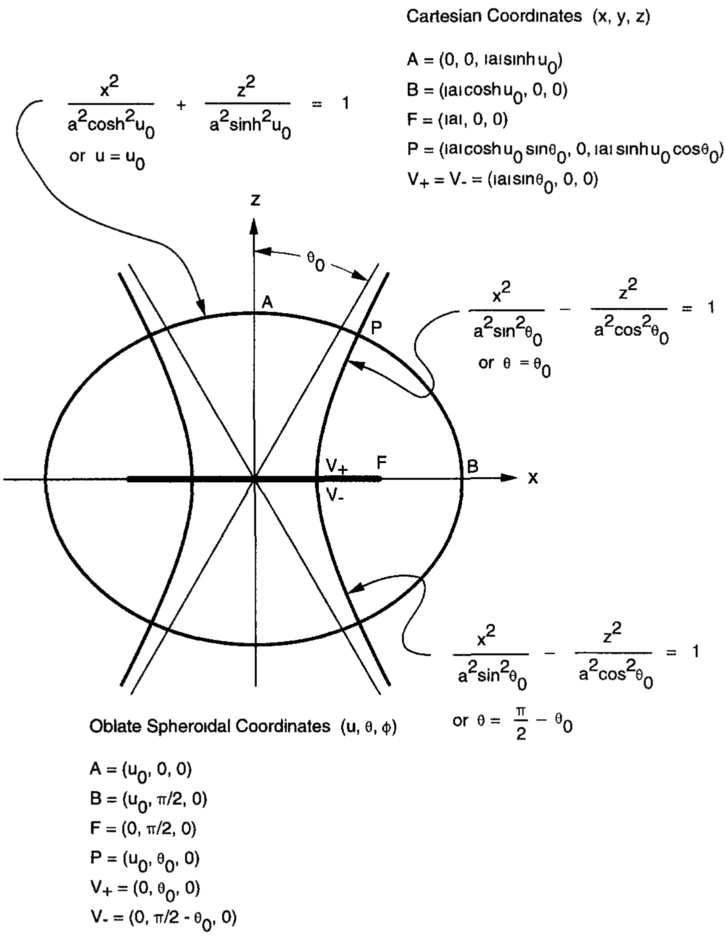


Fig. 9.2. The coordinates of some geometrically significant points.

and

$$\sigma = -a \cos \theta. \tag{9.98b}$$

To get the metric for the flat space, we note that

$$\begin{aligned} s &= \hat{\gamma}_1 x + \hat{\gamma}_2 y + \hat{\gamma}_3 z \\ &= \hat{\gamma}_1 |a| \cosh u \sin \theta \cos \phi + \hat{\gamma}_2 |a| \cosh u \sin \theta \sin \phi \\ &\quad + \hat{\gamma}_3 |a| \sinh u \cos \theta \end{aligned}$$

and thus

$$\hat{\gamma}_u = \frac{\partial s}{\partial u} = |a|[\hat{\gamma}_1 \sinh u \sin \theta \cos \phi + \hat{\gamma}_2 \sinh u \sin \theta \sin \phi + \hat{\gamma}_3 \cosh u \cos \theta].$$

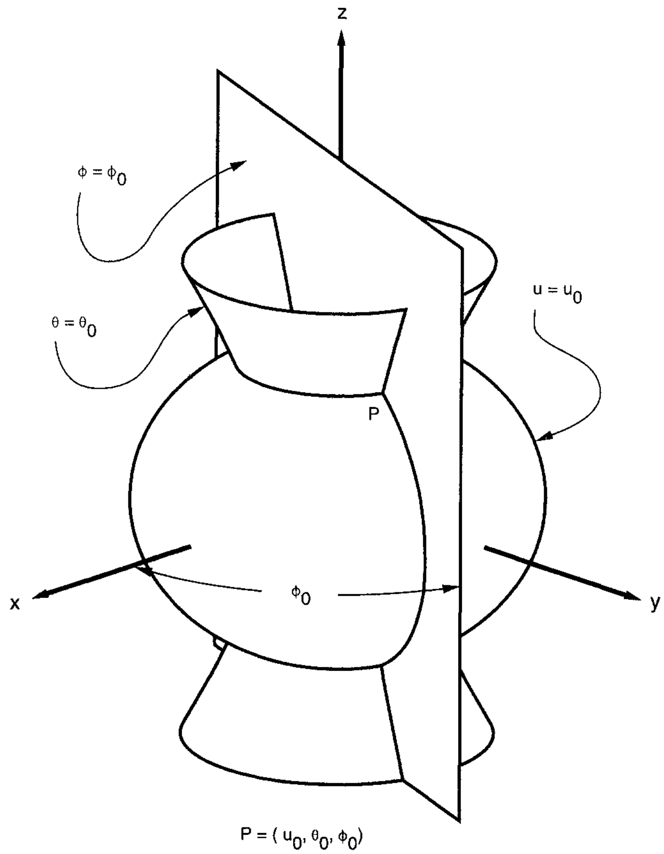


Fig. 9.3. The location of a point P is defined by the intersection of three surfaces: an oblate spheroid, a single sheeted hyperboloid, and a plane passing through the z-axis.

$$\hat{\gamma}_\theta = |a|[\hat{\gamma}_1 \cosh u \cos\theta \cos\phi + \hat{\gamma}_2 \cosh u \cos\theta \sin\phi - \hat{\gamma}_3 \sinh u \sin\theta].$$

and

$$\hat{\gamma}_\phi = |a|[-\hat{\gamma}_1 \cosh u \sin\theta \sin\phi + \hat{\gamma}_2 \cosh u \sin\theta \cos\phi].$$

If we retain the $(-, -, -)$ signature for the three space components then

$$\begin{bmatrix} g_{tt} & g_{tu} & g_{t\theta} & g_{t\phi} \\ g_{ut} & g_{uu} & g_{u\theta} & g_{u\phi} \\ g_{\theta t} & g_{\theta u} & g_{\theta\theta} & g_{\theta\phi} \\ g_{\phi t} & g_{\phi u} & g_{\phi\theta} & g_{\phi\phi} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -a^2(\sinh^2 u + \cos^2\theta) & 0 & 0 \\ 0 & 0 & -a^2(\sinh^2 u + \cos^2\theta) & 0 \\ 0 & 0 & 0 & -a^2\cosh^2 u \cdot \sin^2\theta \end{bmatrix} \tag{9.99}$$

To compute

$$\boldsymbol{\lambda} = -\frac{(I + \hat{J}\mathbf{d}\sigma)\mathbf{d}\rho}{\langle \mathbf{d}\rho, \mathbf{d}\rho \rangle}, \tag{9.100}$$

we note that

$$\mathbf{d}\rho = \left(\gamma^u \frac{\partial}{\partial u} + \gamma^\theta \frac{\partial}{\partial \theta} + \gamma^\phi \frac{\partial}{\partial \phi}\right)|a| \sinh u = \gamma^u |a| \cosh u. \tag{9.101}$$

Similarly

$$\mathbf{d}\sigma = \mathbf{d}(-a \cos \theta) = \gamma^\theta a \sin \theta. \tag{9.102}$$

Also

$$\langle \mathbf{d}\rho, \mathbf{d}\rho \rangle = g^{uu} a^2 \cosh^2 u = \frac{-\cosh^2 u}{\sinh^2 u + \cos^2 \theta}. \tag{9.103}$$

Furthermore

$$\hat{J} = \frac{\gamma_u \gamma_\theta \gamma_\phi}{|a|^3(\sinh^2 u + \cos^2 \theta)(\cosh u \cdot \sin \theta)}. \tag{9.104}$$

In addition

$$\begin{aligned} \hat{J}\mathbf{d}\sigma\mathbf{d}\rho &= \frac{\gamma_\phi}{a(\sinh^2 u + \cos^2 \theta)} = \frac{g_{\phi\phi}}{a(\sinh^2 u + \cos^2 \theta)} \gamma^\phi \\ &= -\frac{a \cosh^2 u \cdot \sin^2 \theta}{(\sinh^2 u + \cos^2 \theta)} \gamma^\phi. \end{aligned} \tag{9.105}$$

Putting together Eqs. (9.101), (9.103), and (9.105), Eq. (9.100) becomes

$$\boldsymbol{\lambda} = \frac{|a|(\sinh^2 u + \cos^2 \theta)}{\cosh u} \gamma^u - a \sin^2 \theta \gamma^\phi. \tag{9.106}$$

To obtain an expression for w_0, we note that

$$\begin{aligned} \alpha + \mathrm{i}\beta &= \frac{1}{\rho + \mathrm{i}\sigma} = \frac{1}{|a| \sinh u - \mathrm{i}a \cos \theta} \\ &= \frac{|a| \sinh u + \mathrm{i}a \cos \theta}{a^2(\sinh^2 u + \cos^2 \theta)}. \end{aligned}$$

Thus

$$\alpha = \frac{\sinh u}{|a|(\sinh^2 u + \cos^2 \theta)}.$$

Therefore

$$(w_0, w_u, w_\theta, w_\phi) = w_0\left(1, \frac{|a|(\sinh^2 u + \cos^2 \theta)}{\cosh u}, 0, -a \sin^2 \theta\right), \quad (9.107)$$

where

$$w_0 = \left[\frac{\sinh u}{|a|(\sinh^2 u + \cos^2 \theta)}\right]^{\frac{1}{2}}. \quad (9.108)$$

The null vector field $\boldsymbol{w}$ lends itself to a simple geometric interpretation. To pursue this interpretation, it is useful to obtain the upper index components. In Cartesian coordinates, we were able to use the flat space metric and its inverse to lower and raise the indices of $\boldsymbol{w}$. This property of $\boldsymbol{w}$ must carry over to alternate coordinate systems. Thus we can use the inverse of the flat space metric of Eq. (9.99) to raise the indices of $\boldsymbol{w}$. Doing this, we get

$$(w^0, w^u, w^\theta, w^\phi) = \alpha^{\frac{1}{2}}\left(1, -\frac{1}{|a| \cosh u}, 0, \frac{1}{a \cosh^2 u}\right). \quad (9.109)$$

A null curve following the flow of this vector field must satisfy the set of equations:

$$\frac{\mathrm{d}t}{\mathrm{d}s} = f, \quad \frac{\mathrm{d}u}{\mathrm{d}s} = \frac{-f}{|a| \cosh u}, \quad \frac{\mathrm{d}\theta}{\mathrm{d}s} = 0, \quad \text{and} \quad \frac{\mathrm{d}\phi}{\mathrm{d}s} = \frac{f}{a \cosh^2 u}, \quad (9.110)$$

where f is any continuous positive definite function of the coordinates.

Since $\mathrm{d}\theta/\mathrm{d}s = 0$, any given member of this family of null curves must adhere to the surface of the hyperboloid corresponding to that fixed value of θ.

It is well known that a hyperboloid is a ruled surface. It was observed by Boyer and Lindquist (1967) that the flow lines of the $\boldsymbol{w}$ field are the rulings of the coordinate hyperboloids shown in Fig. 9.4. To substantiate this, we merely note that not only do the flow lines of the $\boldsymbol{w}$ field adhere to the surface of the coordinate hyperboloids but they are straight lines in the flat metric. (See Problem 9.2.)

Pairs of rulings cross at the waist of each coordinate hyperboloid. (See Fig. 9.5.) Examining the signs of Eqs. (9.110), we see that if a is positive, the null geodesics correspond to the rulings shown in Fig. 9.3 with the future direction toward the source. If a is negative, the null geodesics correspond to the alternate set of rulings with the future direction also toward the source.

One must be cautious about reading too much into these figures. What is a straight line in the flat space metric may not be a geodesic or "straight line" in the curved space of the Kerr metric. The $\boldsymbol{w}$ field corresponds to flow lines which are geodesics in both the flat space metric and the Kerr metric.

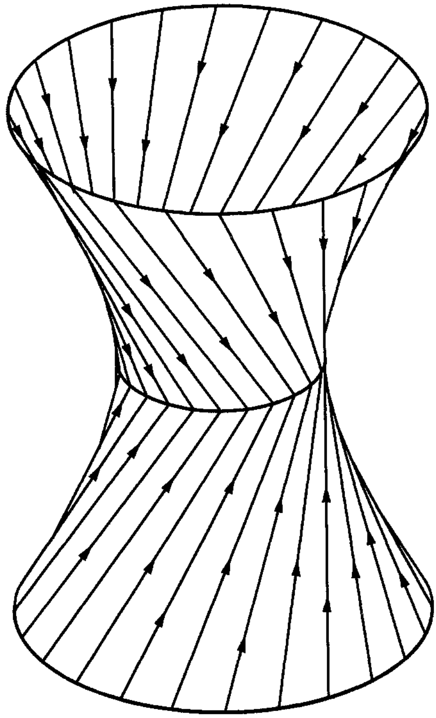


Fig. 9.4. Members of the null vector field $\boldsymbol{w}$ are straight lines on the surfaces of single sheeted hyperboloids.

However, there is no reason to believe that the alternate set of hyperboloid rulings correspond to geodesics in the curved space. In Problem 9.4, it is shown that in the Kerr metric there is a family of geodesics which adhere to the surface of the coordinate hyperboloids but which are clearly not straight lines in the flat space metric.

Now let us turn to the problem of establishing a physical interpretation for a. Combining the flat space metric of Eq. (9.99) with the computed components of $\boldsymbol{w}$ in Eq. (9.107) in the manner indicated by Eq. (9.4), we have the line element for the Kerr metric:

$$\begin{aligned}(\mathrm{d}s)^2 &= (\mathrm{d}t)^2 - a^2(\sinh^2 u + \cos^2 \theta)[(\mathrm{d}u)^2 + (\mathrm{d}\theta)^2] - a^2 \cosh^2 u \sin^2 \theta (\mathrm{d}\phi)^2 \\ &\quad - \frac{2m \sinh u}{|a|(\sinh^2 u + \cos^2 \theta)} \\ &\quad \times \left[\mathrm{d}t + \frac{|a|(\sinh^2 u + \cos^2 \theta)}{\cosh u}\,\mathrm{d}u - a \sin^2 \theta \,\mathrm{d}\phi\right]^2 . \end{aligned} \tag{9.111}$$

In most presentations $\rho = |a| \sinh u$ instead of u is used to label a particular oblate spheroid. The variable ρ equals half the length of the axis

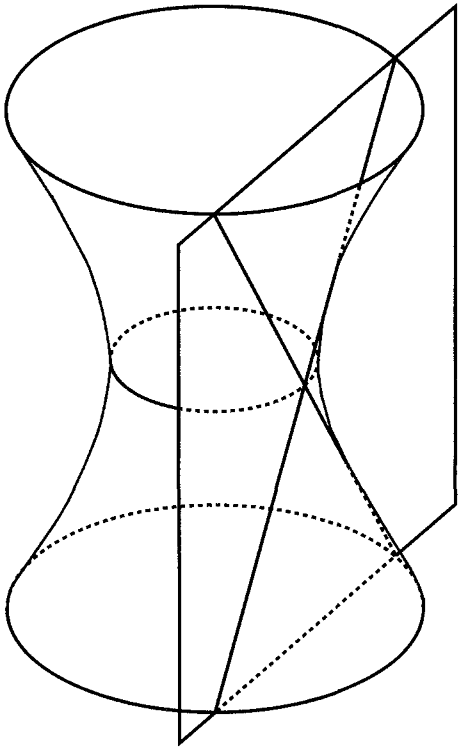


Fig. 9.5. The intersection of a single sheeted hyperboloid and a plane tangent to the hyperboloid at the waist is a pair of straight lines.

of rotation for the labeled oblate spheroid. (Half the major axis of the rotated ellipse is $\sqrt{\rho^2 + a^2}$.) Using this parameter, the line element becomes:

$$(\mathrm{d}s)^2 = (\mathrm{d}t)^2 - (\rho^2 + a^2 \cos^2 \theta)\left[\frac{(\mathrm{d}\rho)^2}{a^2 + \rho^2} + (\mathrm{d}\theta)^2\right] - (\rho^2 + a^2) \sin^2 \theta (\mathrm{d}\phi)^2$$
$$- \frac{2m\rho}{\rho^2 + a^2 \cos^2 \theta}\left[\mathrm{d}t + \frac{\rho^2 + a^2 \cos^2 \theta}{\rho^2 + a^2}\mathrm{d}\rho - a \sin^2 \theta \,\mathrm{d}\phi\right]^2. \qquad (9.112)$$

The line element of Eq. (9.111) has three off-diagonal terms: $\mathrm{d}t\,\mathrm{d}u$, $\mathrm{d}t\,\mathrm{d}\phi$, and $\mathrm{d}u\,\mathrm{d}\phi$. It is possible to eliminate $\mathrm{d}t\,\mathrm{d}u$, and $\mathrm{d}u\,\mathrm{d}\phi$ by a coordinate transformation. In particular, instead of measuring the angle of longitude ϕ from the xz-plane, it is useful to measure it from a surface through the z-axis whose location depends on u (or ρ). In a similar spirit it is useful to reset our clocks on each oblate spheroid. Doing this we have,

$$t = \hat{t} + A(u), \qquad (9.113a)$$

$$\phi = \hat{\phi} + B(u), \qquad (9.113b)$$

$$\mathrm{d}t = \mathrm{d}\hat{t} + A'(u)\,\mathrm{d}u, \tag{9.114a}$$

$$\mathrm{d}\phi = \mathrm{d}\hat{\phi} + B'(u)\,\mathrm{d}u. \tag{9.114b}$$

When one substitutes these formulas for $\mathrm{d}t$ and $\mathrm{d}\phi$ into the line element of Eq. (9.111) and then sets the coefficients of $\mathrm{d}t\,\mathrm{d}u$ and $\mathrm{d}\phi\,\mathrm{d}u$ equal to zero, one obtains two linear equations for $A'(u)$ and $B'(u)$. When these equations are solved, one gets

$$\begin{aligned} A'(u)\,\mathrm{d}u &= \frac{2m|a| \sinh u \cosh u}{(|a|\cosh^2 u - 2m \sinh u)}\,\mathrm{d}u \\ &= \frac{2m\rho}{\rho^2 + a^2 - 2m\rho}\,\mathrm{d}\rho \end{aligned} \tag{9.115a}$$

and

$$\begin{aligned} B'(u)\,\mathrm{d}u &= \frac{2ma \sinh u}{|a|\cosh u(|a|\cosh^2 u - 2m\sinh u)}\,\mathrm{d}u \\ &= \frac{2ma\rho}{(\rho^2 + a^2)(\rho^2 + a^2 - 2m\rho)}\,\mathrm{d}\rho. \end{aligned} \tag{9.115b}$$

These equations can be integrated with respect to ρ by the method of partial fractions but it is a nuisance to do so because one must consider three cases: $|a| < m$, $|a| = m$, and $|a| > m$. However, to get the line element, it is unnecessary to carry out the integrations.

Using Eqs. (9.113a and b) along with Eqs. (9.114a and b), one obtains

$$\begin{aligned} (\mathrm{d}s)^2 = {} & \left[1 - \frac{2m \sinh u}{|a|(\sinh^2 u + \cos^2\theta)}\right](\mathrm{d}\hat{t})^2 \\ & - \frac{|a|^3(\sinh^2 u + \cos^2\theta)\cosh^2 u}{(|a|\cosh^2 u - 2m\sinh u)}(\mathrm{d}u)^2 \\ & - a^2(\sinh^2 u + \cos^2\theta)(\mathrm{d}\theta)^2 \\ & - \left[a^2\cosh^2 u \sin^2\theta + \frac{2|a|m \sinh u \sin^4\theta}{\sinh^2 u + \cos^2\theta}\right](\mathrm{d}\hat{\phi})^2 \\ & + \frac{4ma \sinh u \sin^2\theta}{|a|(\sinh^2 u + \cos^2\theta)}\,\mathrm{d}\hat{\phi}\,\mathrm{d}\hat{t} \end{aligned} \tag{9.116}$$

equivalently

$$(\mathrm{d}s)^2 = \left(1 - \frac{2m\rho}{(\rho^2 + a^2\cos^2\theta)}\right)(\mathrm{d}\hat{t})^2 - \frac{(\rho^2 + a^2\cos^2\theta)}{\rho^2 + a^2 - 2m\rho}(\mathrm{d}\rho)^2 - (\rho^2 + a^2\cos^2\theta)(\mathrm{d}\theta)^2 - \left[(\rho^2 + a^2)\sin^2\theta + \frac{2a^2 m\rho\sin^4\theta}{\rho^2 + a^2\cos^2\theta}\right](\mathrm{d}\hat{\phi})^2 + \frac{4am\rho\sin^2\theta}{\rho^2 + a^2\cos^2\theta}\,\mathrm{d}\hat{\phi}\,\mathrm{d}\hat{t}. \tag{9.117}$$

Except for a sign discrepancy in the off-diagonal term, the coordinate system of Eq. (9.117) is one of several discussed in a paper by Robert Boyer and Richard Lindquist in 1967. Since that time, these parameters have become known as the *Boyer–Lindquist coordinates*. This coordinate system enables us to obtain a physical interpretation for a.

In 1918, Von J. Lense and Hans Thirring published an approximate solution to Einstein's field equations for a slowly rotating spherical source with a weak field. Their computed line element was

$$(\mathrm{d}s)^2 = \left(1 - \frac{2kM}{r}\right)(\mathrm{d}t)^2 - \left(1 + \frac{2kM}{r}\right)[(\mathrm{d}x)^2 + (\mathrm{d}y)^2 + (\mathrm{d}z)^2] + \frac{4kJ}{r}\sin^2\theta\,\mathrm{d}\phi\,\mathrm{d}t, \tag{9.118}$$

where k is the gravitational constant, M is the mass of the source, and J is the angular momentum of the source.

The line element of Lense and Thirring can be obtained from Eq. (9.117) by approximation. We first expand the coefficients in Eq. (9.117) in powers of (a/ρ) and retain only the zero and first-order terms. Doing this we get:

$$(\mathrm{d}s)^2 = \left(1 - \frac{2m}{\rho}\right)(\mathrm{d}\hat{t})^2 - \left(1 - \frac{2m}{\rho}\right)^{-1}(\mathrm{d}\rho)^2 - \rho^2((\mathrm{d}\theta)^2 + \sin^2\theta(\mathrm{d}\hat{\phi})^2) + \frac{4am\sin^2\theta}{\rho}\,\mathrm{d}\hat{\phi}\,\mathrm{d}\hat{t}. \tag{9.119}$$

We now substitute $\rho = r + m$ and then expand the terms of the resulting line element in powers of m/r. If we retain only the zero and first-order terms,

we have

$$(ds)^2 = \left(1 - \frac{2m}{r}\right)(d\hat{t})^2 - \left(1 + \frac{2m}{r}\right)[(dr)^2 + r^2(d\theta)^2 + r^2 \sin^2\theta(d\hat{\phi})^2]$$
$$+ \frac{4am}{r}\sin^2\theta\, d\hat{\phi}\, d\hat{t}. \tag{9.120}$$

Comparing this equation with that of Lense and Thirring, we see that they match if we insist on two identifications. One is the usual one of matching kM with the geometric mass m. The other one is

$$Ma = J. \tag{9.121}$$

Thus "a" may be interpreted as the angular momentum of the source per unit mass of the source.

It should be noted that there is a certain amount of arbitrariness in the sign convention. Had we chosen to use the complex conjugate on the right-hand side of Eq. (9.94) for the solution of Laplace's equation, we would have ended up with negative signs in the off-diagomal terms in Eqs. (9.116) and (9.117). In turn, this would lead to the interpretation of "a" as minus the angular momentum per unit mass.

The sign convention used in this text is that used by Subrahmanyan Chandrasekhar in his text, *The Mathematical Theory of Black Holes* (1983). However, it is opposite to that originally used by Roy Kerr (1963).

Problem 9.1. Verify Eq. (9.52). Hint: write in component form and use the fact that $\gamma^i\gamma^{jk} = \gamma^{ijk} + g^{ij}\gamma^k - g^{ik}\gamma^j$. Also verify Eqs. (9.53) and (9.54).

Problem 9.2. Carry out a variation calculation on

$$\int (n_{ij} - 2mw_i w_j)\frac{dx^i}{ds}\frac{dx^j}{ds}\,ds$$

to obtain the set of equations for the geodesics for the Kerr metric. Then show that if

$$\frac{dx^i}{ds} = fw^i,$$

the system reduces to

$$\frac{d}{ds}(fw_k) = 0. \tag{9.122}$$

Show that this equation is satisfied if $df/ds = Af^2$. (You may wish to

use Eq. (9.30).) Note: Eq. (9.122) may also be written as

$$\frac{d^2}{ds^2} x^k = 0.$$

This means that the null geodesics that follow the flow of the w field are straight lines in the flat metric.

Problem 9.3. Show that aside from members of the w field, a vector which is null in the flat metric is not null in the curved space of the Kerr metric.

Problem 9.4. The w field is known as a "principal null congruence." The word "congruence" is used to indicate that the w field generates a space filling system of curves which do not intersect one another. The word "principal" is associated with the role that w plays in the structure of the metric. From symmetry arguments, it was known by Kerr that a second principal congruence of null geodesics had to exist.

One might suspect that this alternate set of geodesics might be the alternate set of rulings for the single sheeted hyperboloids. However, this is not quite true. For one thing, although these rulings are straight lines in the flat metric, we have no reason to believe they are geodesics in the curved metric.

To determine this alternate congruence, we can take advantage of a symmetry in the Kerr metric of Eq. (9.117) expressed in the coordinates of Boyer and Lindquist. It is invariant under the replacement of $d\rho$ by $-d\rho$. (Alternatively we can change the signs of both $d\hat{\phi}$ and $d\hat{t}$.) This means that if we obtain the upper index components of w in the Boyer–Lindquist coordinate system, one can change the sign of the w_ρ component to obtain an alternate null congruence.

(1) Use Eqs. (9.109), (9.114a and b) along with (9.115a and b) to determine that for the Boyer–Lindquist coordinate system:

$$(w^{\hat{t}}, w^{\rho}, w^{\theta}, w^{\hat{\phi}}) = \alpha^{\frac{1}{2}}\left[\frac{\rho^2 + a^2}{\rho^2 + a^2 - 2m\rho}, -1, 0, \frac{a}{\rho^2 + a^2 - 2m\rho}\right].$$

(2) Let $\boldsymbol{k} = (w^{\hat{t}}, w^{\rho}, -w^{\theta}, w^{\hat{\phi}})$ and then use Eqs. (9.114a and b) along with (9.115a and b) to show that in the coordinate system of Eq. (9.112)

$$\begin{aligned} \boldsymbol{k} &= (k^t, k^\rho, k^\theta, k^\phi) \\ &= \alpha^{\frac{1}{2}}\left[\frac{\rho^2 + a^2 + 2m\rho}{\rho^2 + a^2 - 2m\rho}, 1, 0, \frac{a(\rho^2 + a^2 + 2m\rho)}{(\rho^2 + a^2)(\rho^2 + a^2 - 2m\rho)}\right]. \end{aligned}$$

This means that for the coordinate system of Eq. (9.111), where

$\rho = |a| \sinh u$:

$$\boldsymbol{k} = (k^t, k^u, k^\theta, k^\phi) = \alpha^{\frac{1}{2}}\left(\frac{P}{Q}, \frac{1}{|a| \cosh u}, 0, \frac{P}{Qa \cosh^2 u}\right)$$

where $P|a| \cosh^2 u + 2m \sinh u$ and $Q = |a| \cosh^2 u - 2m \sinh u$.

(3) Compare this last equation with Eq. (9.109) and show that when $u \to +\infty$, or $m \to 0$, the flow lines, represented by the vector field $\boldsymbol{k}$, asymptotically approach the alternate set of hyperboloid rulings with an outgoing direction.

Students of black holes are usually more interested in the behavior of the null vector field near the source. If one approaches a source for which $m \geqslant |a|$ then one encounters an apparent singularity on the oblate spheroid defined by the equation $\rho = m + \sqrt{m^2 - a^2}$ where $\rho^2 + a^2 - 2m\rho = 0$. This is a coordinate singularity that can be removed by simply factoring out $(\rho^2 + a^2 - 2m\rho)^{-1}$ and renormalizing the vector field $\boldsymbol{k}$. Nonetheless, the surface of this oblate spheroid is physically interesting.

For one thing, a sign reversal occurs when one crosses the surface. Outside the surface, $\boldsymbol{k}$ represents an outgoing field of null geodesics. Just inside the surface, the field becomes ingoing.

(4) Show that when $\rho \to m + \sqrt{m^2 - a^2}$, the vector field $(k^t, k^\rho, k^\theta, k^\phi)$ approaches some scalar multiple of $(\rho^2 + a^2, 0, 0, a)$. Since $k^\rho = k^\theta = 0$, these members of the $\boldsymbol{k}$ congruence correspond to unstable circular orbits for photons parallel to the xy-plane formed by the intersection of various hyperboloids with the oblate spheroid defined by the equation $\rho = m + \sqrt{m^2 - a^2}$.

Further physical consequences of this surface and other surfaces near a black hole are discussed in almost any current text on general relativity.

10

PETROV'S CANONICAL FORMS FOR THE WEYL TENSOR AND ANOTHER APPROACH TO THE KERR METRIC

10.1 Petrov's Canonical Forms for the Weyl Tensor

In seeking out solutions for Einstein's field equations for empty space, one can save considerable labor by imposing symmetry conditions on the Riemann tensor. These conditions are particularly useful if they also guarantee that the Ricci tensor is zero. Aleksei Zinoveivich Petrov (1954, 1969) devised a classification scheme that divides all Riemann tensors into six classes.

Users of the formalism of spinors have successfully exploited the use of these symmetry classes to construct a substantial number of solutions to Einstein's field equations. Much of this work was summarized in *Exact Solutions of Einstein's Field Equations* by Kramer *et al.* (1980). However, these same symmetries can be exploited with the formalism of Clifford algebra.

At the end of this section, it is shown how symmetries imposed by the Petrov classification can expedite calculation of the Schwarzschild metric in a rather trivial manner. Section 10.3 in this chapter is devoted to showing how the Petrov symmetries, along with the Bianchi identities, can be used in a somewhat different manner to obtain the Kerr metric.

This section is devoted to determining the canonical representation for each of the six Petrov classes. The next section is devoted to finding the principal null directions for each of these canonical representations. In the third and last section of this chapter, we adjust one of Petrov's canonical forms so that one of the principal null directions conforms with the null vector that appears in the line element of the Kerr metric. The resulting symmetries imposed on the Riemann curvature 2-forms are then exploited to obtain the Kerr metric in a straightforward manner.

The Petrov classification scheme for the Riemann tensor R_{ijkm} may also be considered a classification scheme for the *Weyl* or *conformal tensor* C_{ijkm}

where C_{ijkm} is defined by the equation:

$$C_{ijkm} = R_{ijkm} - R_{i[k}g_{m]j} + R_{j[k}g_{m]i} + \tfrac{1}{3}Rg_{i[k}g_{m]j}. \tag{10.1}$$

The square brackets in Eq. (10.1) indicate that an antisymmetry is imposed on the indices m and k. In particular:

$$\begin{aligned} C_{ijkm} = R_{ijkm} &- \tfrac{1}{2}(R_{ik}g_{mj} - R_{im}g_{kj}) \\ &+ \tfrac{1}{2}(R_{jk}g_{mi} - R_{jm}g_{ki}) + \tfrac{1}{6}R(g_{ik}g_{mj} - g_{im}g_{kj}). \end{aligned}$$

We note that when the Ricci tensor R_{ij} is zero, $C_{ijkm} = R_{ijkm}$. The reader should check the claim that

$$C_{ijk}{}^{j} = 0. \tag{10.2}$$

Because of Eq. (10.2), the Weyl tensor is said to be the "trace free part" of the Riemann tensor. The reader should also check that the Weyl tensor has the same symmetries as the Riemann tensor, that is

$$C_{ijkm} = C_{[ij]km} \tag{10.3}$$

and

$$C_{i[jkm]} = 0. \tag{10.4}$$

To analyze the Petrov classes, it is useful to use an orthonormal noncoordinate frame; that is

$$\hat{\gamma}_j\hat{\gamma}_k + \hat{\gamma}_k\hat{\gamma}_j = 2n_{jk}. \tag{10.5}$$

The Weyl tensor will be treated as a matrix. That is, if $\boldsymbol{a} = \frac{1}{2}a^{ij}\hat{\gamma}_{ij}$ and $b = \frac{1}{2}b^{ij}\hat{\gamma}_{ij}$, then $\boldsymbol{b} = C\boldsymbol{a}$ if

$$b^{ij} = C^{ij}{}_{23}a^{23} + C^{ij}{}_{31}a^{31} + C^{ij}{}_{12}a^{12} + C^{ij}{}_{10}a^{10} + C^{ij}{}_{20}a^{20} + C^{ij}{}_{30}a^{30}. \tag{10.6}$$

Writing C as a matrix, we have

$$C = \begin{bmatrix} C^{23}{}_{23} & C^{23}{}_{31} & C^{23}{}_{12} & C^{23}{}_{10} & C^{23}{}_{20} & C^{23}{}_{30} \\ C^{31}{}_{23} & C^{31}{}_{31} & C^{31}{}_{12} & C^{31}{}_{10} & C^{31}{}_{20} & C^{31}{}_{30} \\ C^{12}{}_{23} & C^{12}{}_{31} & C^{12}{}_{12} & C^{12}{}_{10} & C^{12}{}_{20} & C^{12}{}_{30} \\ C^{10}{}_{23} & C^{10}{}_{31} & C^{10}{}_{12} & C^{10}{}_{10} & C^{10}{}_{20} & C^{10}{}_{30} \\ C^{20}{}_{23} & C^{20}{}_{31} & C^{20}{}_{12} & C^{20}{}_{10} & C^{20}{}_{20} & C^{20}{}_{30} \\ C^{30}{}_{23} & C^{30}{}_{31} & C^{30}{}_{12} & C^{30}{}_{10} & C^{30}{}_{20} & C^{30}{}_{30} \end{bmatrix}. \tag{10.7}$$

For a Weyl matrix there is a substantial amount of symmetry among the components. For example, since $C^{i}{}_{j} = 0$, it follows that

$$C^{1}{}_{1} = C^{1k}{}_{1k} = C^{10}{}_{10} + C^{12}{}_{12} + C^{13}{}_{13} = 0$$

or

$$C^{10}{}_{10} + C^{12}{}_{12} + C^{31}{}_{31} = 0. \tag{10.8}$$

Similarly

$$C^{2}{}_{2} = C^{20}{}_{20} + C^{12}{}_{12} + C^{23}{}_{23} = 0, \tag{10.9}$$

$$C^{3}{}_{3} = C^{30}{}_{30} + C^{31}{}_{31} + C^{23}{}_{23} = 0, \tag{10.10}$$

$$C^{0}{}_{0} = C^{10}{}_{10} + C^{20}{}_{20} + C^{30}{}_{30} = 0. \tag{10.11}$$

Adding Eqs. (10.8)–(10.10), we have

$$2(C^{10}{}_{10} + C^{20}{}_{20} + C^{30}{}_{30}) + (C^{23}{}_{23} + C^{31}{}_{31} + C^{12}{}_{12}) = 0.$$

Combining this equation with Eq. (10.11), it is clear that

$$C^{23}{}_{23} + C^{31}{}_{31} + C^{12}{}_{12} = 0. \tag{10.12}$$

Subtracting Eq. (10.12) from Eq. (10.8) gives us

$$C^{10}{}_{10} = C^{23}{}_{23}. \tag{10.13}$$

Similarly

$$C^{20}{}_{20} = C^{31}{}_{31} \tag{10.14}$$

and

$$C^{30}{}_{30} = C^{12}{}_{12}. \tag{10.15}$$

Thus we see that the diagonal entries in the 3×3 block in the lower right-hand corner of the matrix in Eq. (10.7) are identical to the corresponding diagonal entries in the upper left-hand corner of matrix C. Also from either Eq. (10.10) or Eq. (10.12), we see that the traces of these 3×3 blocks are zero. In actuality not only are the diagonal entries of these 3×3 blocks identical but the entire 3×3 blocks are identical to one another and symmetric.

To see that the two blocks are symmetric, we note for example that

$$C^{12}{}_{31} = C_{1231} = C_{3112} = C^{31}{}_{12}$$

and

$$C^{20}{}_{30} = -C_{2030} = -C_{3020} = C^{30}{}_{20}.$$

The identity of the two blocks follows from the fact that $C^j{}_k = 0$ for $j, k = 1, 2$, or 3 and $j \neq k$. For example

$$C^2{}_3 = 0 = C^{20}{}_{30} + C^{21}{}_{31}$$

or

$$C^{20}{}_{30} = C^{12}{}_{31}.$$

Now consider the remaining 3×3 blocks. As will be shown, these two blocks are also traceless and symmetric. They are not equal to one another but one is the additive inverse of the other.

To show that the trace of the upper right-hand block is zero, we note that

$$3C_{[231]0} = 0 = C_{2310} + C_{3120} + C_{1230} = C^{23}{}_{10} + C^{31}{}_{20} + C^{12}{}_{30}.$$

The fact that the 3×3 block in the upper right-hand corner is symmetric follows from the fact that $C^k{}_0 = 0$ for $k = 1, 2$, and 3. For example

$$C^1{}_0 = 0 = C^{12}{}_{02} + C^{13}{}_{03} = -C^{12}{}_{20} + C^{31}{}_{30}$$

or

$$C^{12}{}_{20} = C^{31}{}_{30}.$$

To show that, except for a sign reversal, the 3×3 block in the upper right-hand corner is equal to the 3×3 block in the lower left-hand corner, we observe that for $i, j, k = 1, 2$, or 3

$$C^{jk}{}_{i0} = C_{jki0} = C_{i0jk} = -C^{i0}{}_{jk}.$$

To summarize

$$C = \begin{bmatrix} M & N \\ -N & M \end{bmatrix} \tag{10.16}$$

where both M and N are traceless and symmetric 3×3 matrices. One is tempted to say that this form of C represents the most general form of a Weyl matrix or a matrix representing the curvature for empty space using an orthonormal noncoordinate basis. Actually this is almost but not quite true. The Weyl matrix must also be consistent with the Bianchi identities. (See Appendix A.3.)

Nonetheless if we replace the Weyl matrix C on the left-hand side of Eq. (10.16) by the corresponding matrix for the Riemann tensor, the resulting matrix equation may be regarded as the complete set of Einstein's field equations for empty space. In this context it is assumed that the Riemann tensor matrix used to replace the Weyl tensor matrix is derived from

some generalized metric. This guarantees that the Bianchi identities are satisfied.

At the end of this section it is shown how this form of Einstein's field equations can be used to expedite the solution of the Schwarschild metric.

To study the Petrov classes, it is useful to modify Eq. (10.6). In particular it is useful to replace the equation

$$\begin{bmatrix} M & N \\ -N & M \end{bmatrix} \begin{bmatrix} a^{23} \\ a^{31} \\ a^{12} \\ a^{10} \\ a^{20} \\ a^{30} \end{bmatrix} = \begin{bmatrix} b^{23} \\ b^{31} \\ b^{12} \\ b^{10} \\ b^{20} \\ b^{30} \end{bmatrix} \tag{10.17}$$

by the equation

$$\frac{1}{2}\begin{bmatrix} I & -\mathrm{i}I \\ -\mathrm{i}I & I \end{bmatrix}\begin{bmatrix} M & N \\ -N & M \end{bmatrix}\begin{bmatrix} I & \mathrm{i}I \\ \mathrm{i}I & I \end{bmatrix}\begin{bmatrix} I & -\mathrm{i}I \\ -\mathrm{i}I & I \end{bmatrix}\begin{bmatrix} a^{23} \\ a^{31} \\ a^{12} \\ a^{10} \\ a^{20} \\ a^{30} \end{bmatrix}$$

$$= \begin{bmatrix} I & -\mathrm{i}I \\ -\mathrm{i}I & I \end{bmatrix}\begin{bmatrix} b^{23} \\ b^{31} \\ b^{12} \\ b^{10} \\ b^{20} \\ b^{30} \end{bmatrix}$$

or

$$\begin{bmatrix} M + \mathrm{i}N & 0 \\ 0 & M - \mathrm{i}N \end{bmatrix}\begin{bmatrix} a^{23} - \mathrm{i}a^{10} \\ a^{31} - \mathrm{i}a^{20} \\ a^{12} - \mathrm{i}a^{30} \\ a^{10} - \mathrm{i}a^{23} \\ a^{20} - \mathrm{i}a^{31} \\ a^{30} - \mathrm{i}a^{12} \end{bmatrix} = \begin{bmatrix} b^{23} - \mathrm{i}b^{10} \\ b^{31} - \mathrm{i}b^{20} \\ b^{12} - \mathrm{i}b^{30} \\ b^{10} - \mathrm{i}b^{23} \\ b^{20} - \mathrm{i}b^{31} \\ b^{30} - \mathrm{i}b^{12} \end{bmatrix}.$$

This last equation may be decomposed into two equations:

$$[M + \mathrm{i}N]\begin{bmatrix} a^{23} - \mathrm{i}a^{10} \\ a^{31} - \mathrm{i}a^{20} \\ a^{12} - \mathrm{i}a^{30} \end{bmatrix} = \begin{bmatrix} b^{23} - \mathrm{i}b^{10} \\ b^{31} - \mathrm{i}b^{20} \\ b^{12} - \mathrm{i}b^{30} \end{bmatrix} \tag{10.18}$$

$$[M - \mathrm{i}N]\begin{bmatrix} a^{23} + \mathrm{i}a^{10} \\ a^{31} + \mathrm{i}a^{20} \\ a^{12} + \mathrm{i}a^{30} \end{bmatrix} = \begin{bmatrix} b^{23} + \mathrm{i}b^{10} \\ b^{31} + \mathrm{i}b^{20} \\ b^{12} + \mathrm{i}b^{30} \end{bmatrix} \tag{10.19}$$

Equation (10.18) may be considered the result of applying the projection operator $\frac{1}{2}(I + \mathrm{i}J)$ to Eq. (10.17). Similarly Eq. (10.19) is the result of applying the projection operator $\frac{1}{2}(I - \mathrm{i}J)$ to the same equation. It is the matrix $M + \mathrm{i}N = P$ that is known as the *Petrov matrix*.

One should ask whether or not it is possible to simplify the Petrov matrix by a Lorentz transformation. Ideally one would like to find a transformation which diagonalizes C. Such a transformation would also diagonalize P. In general, this is not possible. However, one can generally find a Lorentz transformation which will make most of the components of P equal to zero.

To see how the Petrov matrix behaves under various Lorentz transformations, let us see first how it behaves under a rotation in the xy-plane. We note that

$$\begin{aligned}
(\acute{\gamma}^{23} - \mathrm{i}\acute{\gamma}^{10}) &= \left(I\cos\frac{\theta}{2} + \gamma^{12}\sin\frac{\theta}{2}\right)(\gamma^{23} - \mathrm{i}\gamma^{10})\left(I\cos\frac{\theta}{2} - \gamma^{12}\sin\frac{\theta}{2}\right) \\
&= \left(I\cos\frac{\theta}{2} + \gamma^{12}\sin\frac{\theta}{2}\right)(I + \mathrm{i}J)\gamma^{23}\left(I\cos\frac{\theta}{2} - \gamma^{12}\sin\frac{\theta}{2}\right) \\
&= (I + \mathrm{i}J)\gamma^{23}\left(I\cos\frac{\theta}{2} - \gamma^{12}\sin\frac{\theta}{2}\right)\left(I\cos\frac{\theta}{2} - \gamma^{12}\sin\frac{\theta}{2}\right) \\
&= (I + \mathrm{i}J)\gamma^{23}(I\cos\theta - \gamma^{12}\sin\theta) \\
&= (I + \mathrm{i}J)(\gamma^{23}\cos\theta + \gamma^{31}\sin\theta).
\end{aligned}$$

Thus

$$\acute{\gamma}^{23} - \mathrm{i}\acute{\gamma}^{10} = (\gamma^{23} - \mathrm{i}\gamma^{10})\cos\theta + (\gamma^{31} - \mathrm{i}\gamma^{20})\sin\theta.$$

Similarly

$$\acute{\gamma}^{31} - \mathrm{i}\acute{\gamma}^{20} = -(\gamma^{23} - \mathrm{i}\gamma^{10})\sin\theta + (\gamma^{31} - \mathrm{i}\gamma^{20})\cos\theta,$$

while

$$\acute{\gamma}^{12} = \gamma^{12} \quad \text{and} \quad \acute{\gamma}^{30} = \gamma^{30}.$$

The coefficients a^{jk} transform in the same manner as the Clifford numbers $\hat{\gamma}^{jk}$. Thus

$$\vec{a}' = R_z\vec{a}, \tag{10.20}$$

where

$$\vec{a} = \begin{bmatrix} a^{23} - \mathrm{i}a^{10} \\ a^{31} - \mathrm{i}a^{20} \\ a^{12} - \mathrm{i}a^{30} \end{bmatrix} \tag{10.21}$$

and

$$R_z = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{10.22}$$

With this notation, Eq. (10.18) becomes $P\vec{a} = \vec{b}$ which means that $(R_zPR_z^{-1})R_z\vec{a} = R_z\vec{b}$ or $(R_zPR_z^{-1})\vec{a}' = \vec{b}'$. Thus we have

$$P' = R_zPR_z^{-1}. \tag{10.23}$$

In a similar fashion, for a boost in the z-direction

$$\mathscr{B} = I\cosh\frac{\phi}{2} - \hat{\gamma}^{30}\sinh\frac{\phi}{2}$$

and

$$P' = B_zPB_z^{-1} \tag{10.24}$$

where

$$B_z = \begin{bmatrix} \cosh\phi & -\mathrm{i}\sinh\phi & 0 \\ \mathrm{i}\sinh\phi & \cosh\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{10.25}$$

More generally, for any Lorentz transformation $P' = LPL^{-1}$ where L is a member of the group SO(3, $\mathbb{C}$)—that is the group of 3×3 complex matrices with the property that $L \in \mathrm{SO}(3, \mathbb{C}) \Rightarrow L^{-1} = L^{\mathrm{T}}$ (the transpose of L) and $\det L = 1$. This group is a representation of the proper Lorentz group but not a faithful representation. In the group SO(3, $\mathbb{C}$), the transformation which reverses the orientation of all four Dirac matrices ($\gamma^\alpha \to -\gamma^\alpha$) is represented by the identity matrix. The group SO(3, $\mathbb{C}$) is isomorphic to the group of proper orthochronous Lorentz transformations. (*Orthochronous Lorentz*

transformations are those which do not reverse the sign of the time coordinate.)

Now that we have developed some machinery, let us turn to the task of finding canonical forms for the Petrov matrix P. We note that $P^{\mathrm{T}} = P$. Thus when P is real, one can use three orthonormal eigenvectors of P to construct a member of $\mathrm{SO}(3, \mathbb{R})$ to diagonalize P.

Some aspects of this situation remain the same when P is complex. For one thing it is not difficult to show that eigenvectors that correspond to distinct eigenvalues are orthogonal. In particular, if $P\vec{v}_1 = \lambda_1\vec{v}_1$ and $P\vec{v}_2 = \lambda_2\vec{v}_2$, then

$$\lambda_2\vec{v}_1^{\mathrm{T}}\vec{v}_2 = \vec{v}_1^{\mathrm{T}}P\vec{v}_2 = \vec{v}_1^{\mathrm{T}}P^{\mathrm{T}}\vec{v}_2 = (P\vec{v}_1)^{\mathrm{T}}\vec{v}_2 = \lambda_1\vec{v}_1^{\mathrm{T}}\vec{v}_2.$$

Thus $(\lambda_2 - \lambda_1)\vec{v}_1^{\mathrm{T}}\vec{v}_2 = 0$ or $\vec{v}_1^{\mathrm{T}}\vec{v}_2 = 0$ if $\lambda_2 \neq \lambda_1$.

What is different for a complex P is that an eigenvector $\vec{v}$ may be a null vector; that is, it is possible that $\vec{v}^{\mathrm{T}}\vec{v} = 0$. Such an eigenvector cannot be used to construct an orthogonal matrix which might diagonalize P. When P does have such a null eigenvector, P cannot be diagonalized by a Lorentz transformation.

We will now see what can be done. Let us first suppose that P has at least one non-null eigenvector. It is possible to multiply such an eigenvector by a complex scalar so that the resulting vector $\vec{v}$ has the property that $\vec{v}^{\mathrm{T}}\vec{v} = 1$. Any such vector $\vec{v}$ must have the form

$$\vec{v} = \begin{bmatrix} a + ib \\ c + id \\ e + if \end{bmatrix}.$$

Since $\vec{v}^{\mathrm{T}}\vec{v} = 1 > 0$, it is not possible to have $a = c = e = 0$. Clearly there exists a real orthogonal matrix R representing a spatial rotation such that

$$R\begin{bmatrix} a \\ c \\ e \end{bmatrix} = \begin{bmatrix} a' \\ 0 \\ 0 \end{bmatrix} \quad \text{where } a' > 0.$$

Applying the same rotation matrix to the vector $\vec{v}$, we get

$$\vec{v}\,' = R\vec{v} = R\begin{bmatrix} a + ib \\ c + id \\ e + if \end{bmatrix} = \begin{bmatrix} a' + ib' \\ id' \\ if' \end{bmatrix}.$$

Since the rotation matrix preserves the product $\vec{v}^{\mathrm{T}}\vec{v}$, it is clear that $(a' + ib')^2 - (d')^2 - (f')^2 = 1$ and thus $b' = 0$.

An additional rotation in the yz-plane gives us a vector $\vec{v}''$ with the form

$$\vec{v}'' = \begin{bmatrix} a' \\ id'' \\ 0 \end{bmatrix} \qquad \text{where } a' > 0 \quad \text{and} \quad (a')^2 - (d'')^2 = 1.$$

From the form of the boost matrix of Eq. (10.25), it is clear that we can apply a boost operator in the z-direction with $a' = \cosh\phi$ and $d'' = -\sinh\phi$ to obtain

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = B_z \begin{bmatrix} a' \\ id'' \\ 0 \end{bmatrix}.$$

Since P is symmetric, we can write

$$P = \begin{bmatrix} z_1 & z_2 & z_3 \\ z_2 & z_4 & z_5 \\ z_3 & z_5 & z_6 \end{bmatrix}.$$

However, if the rotations and the boost discussed above have been applied to P, we have

$$P \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \lambda_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

and thus

$$P = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & z_4 & z_5 \\ 0 & z_5 & z_6 \end{bmatrix}.$$

If P has a second non-null eigenvector $\vec{w}$ which has been normalized so that $\vec{w}^{\mathrm{T}}\vec{w} = 1$, then $\vec{w}$ can be represented in the form

$$\vec{w} = \begin{bmatrix} 0 \\ a + ib \\ c + id \end{bmatrix} \qquad \text{where } (a)^2 + (c)^2 > 0.$$

Applying a suitable rotation in the yz-plane and a boost in the x-direction

will transform the vector $\vec{w}$ into the form

$$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

When the same Lorentz transformations are applied to P, P must have the form

$$P = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & z_6 \end{bmatrix}.$$

In this situation, it is clear that P has a third non-null eigenvector

$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

with eigenvalue z_6. Thus we see that if P has at least two non-null eigenvectors, it also has a third non-null eigenvector and one can find a noncoordinate orthonormal frame such that

$$P = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \quad \text{where } \lambda_1 + \lambda_2 + \lambda_3 = 0.$$

If none of the eigenvalues are equal, the class of matrices that has this canonical form is known as Petrov's class I.

If two of the eigenvalues are equal but not zero, the class is known as class D. In that case

$$P = \begin{bmatrix} -2\lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}.$$

If all of the eigenvalues are equal, they must be zero. In that case we have a flat space and the class is known as class O.

In Petrov's original classication, class D was considered to be a not very special case of class I. However, the study of principal null directions has shown that it is useful to treat D as a distinct class.

Now let us investigate what was originally labeled Petrov's class II. Members of this class have one nonnull eigenvector but no more. As before there exists a Lorentz transformation that will give the normalized nonnull eigenvector the form

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

When this same Lorentz transformation is applied to the matrix P, it must assume the form

$$P = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & z_1 & z_2 \\ 0 & z_2 & z_3 \end{bmatrix}. \tag{10.26}$$

The 2×2 block in the lower right-hand corner must have a null eigenvector. The corresponding eigenvector for P must have the form

$$\begin{bmatrix} 0 \\ \pm z \\ \mathrm{i}z \end{bmatrix}.$$

No generality is lost if we choose the plus sign. We should note that if such a vector with a minus sign is subjected to a 180° rotation in the xy-plane, one will obtain the version with the plus sign. This same rotation will change the sign of z_2 in Eq. (10.26) but it will otherwise leave the form of P unchanged. We now have

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & z_1 & z_2 \\ 0 & z_2 & z_3 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ \mathrm{i} \end{bmatrix} = \lambda_2 \begin{bmatrix} 0 \\ 1 \\ \mathrm{i} \end{bmatrix}.$$

An immediate consequence of this relation is $z_1 + \mathrm{i}z_2 = \lambda_2$ and $z_2 + \mathrm{i}z_3 = \mathrm{i}\lambda_2$. Using these equations to eliminate z_1 and z_3, we have

$$P = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 - \mathrm{i}z_2 & z_2 \\ 0 & z_2 & \lambda_2 + \mathrm{i}z_2 \end{bmatrix}.$$

Since the trace of P is zero, it follows that

$$P = \begin{bmatrix} -2\lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} + z_2 \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\mathrm{i} & 1 \\ 0 & 1 & \mathrm{i} \end{bmatrix}.$$

Applying a rotation in the yz-plane gives us $P' = R_x P R_x^{\mathrm{T}}$ where

$$R_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix}.$$

Carrying out the matrix multiplication, we find

$$P' = \begin{bmatrix} -2\lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} + z_2\, \mathrm{e}^{\mathrm{i}2\theta} \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\mathrm{i} & 1 \\ 0 & 1 & \mathrm{i} \end{bmatrix}.$$

If we now apply a boost in the x-direction, we have $P'' = B_x P' B_x^{\mathrm{T}}$ where

$$B_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cosh\phi & -\mathrm{i}\sinh\phi \\ 0 & \mathrm{i}\sinh\phi & \cosh\phi \end{bmatrix}.$$

Completing this computation, we have

$$P'' = \begin{bmatrix} -2\lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} + z_2\, \mathrm{e}^{\mathrm{i}2\theta}\, \mathrm{e}^{2\phi} \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\mathrm{i} & 1 \\ 0 & 1 & \mathrm{i} \end{bmatrix}.$$

If $z_2 = 0$, P'' would be a representative of class D. However, by a judicious choice of θ and ϕ, we can equate $z_2 \exp 2(\mathrm{i}\theta + \phi)$ to any nonzero scalar. If we choose that scalar to be i, we have

$$P = \begin{bmatrix} -2\lambda & 0 & 0 \\ 0 & \lambda + 1 & \mathrm{i} \\ 0 & \mathrm{i} & \lambda - 1 \end{bmatrix}.$$

If $\lambda \neq 0$, P is considered to be a representative of class II. If $\lambda = 0$, P is said to be a representative of class N.

Now consider the last Petrov class (class III). If P had more than one eigenvalue, it would have to have at least two eigenvectors. In classes I and D, we covered the case for which there are two nonnull eigenvectors. In classes II and N, we covered the case for which one eigenvector is null and the other is not null. This leaves open the possibility of two null eigenvectors. If the two null eigenvectors had distinct eigenvalues, the two null eigenvectors would be orthogonal. However, that is not possible in a 3-dimensional space. (See Prob. 10.1.) This means that all three eigenvalues must be the same. In turn, this implies that all three eigenvalues must be zero since the trace of P is zero. We will see that if all three eigenvalues are zero and there is no nonnull eigenvector then there must be only one eigenvector which must be null.

As in the cases previously discussed, one can adjust the form of the null eigenvector by a Lorentz transformation. Since the corresponding form of the Petrov matrix P must be compatible, we have

$$P\begin{bmatrix} 0 \\ 1 \\ \mathrm{i} \end{bmatrix} = \begin{bmatrix} z_1 & z_2 & z_3 \\ z_2 & z_4 & z_5 \\ z_3 & z_5 & z_6 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ \mathrm{i} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

From the three component equations, one can eliminate z_3, z_4, and z_6. We then have

$$P = \begin{bmatrix} z_1 & z_2 & \mathrm{i}z_2 \\ z_2 & -\mathrm{i}z_5 & z_5 \\ \mathrm{i}z_2 & z_5 & \mathrm{i}z_5 \end{bmatrix}.$$

Since the trace of P is zero, $z_1 = 0$. We now have $P = z_2 G - \mathrm{i}z_5 G^2$ where

$$G = \begin{bmatrix} 0 & 1 & \mathrm{i} \\ 1 & 0 & 0 \\ \mathrm{i} & 0 & 0 \end{bmatrix} \qquad \text{and} \qquad G^3 = 0.$$

If we now apply the same rotation and the same boost to our class III Petrov matrix that we did to our class II Petrov matrix, we find

$$P = z_2\, \mathrm{e}^{\mathrm{i}\theta + \phi} G - \mathrm{i}z_5\, \mathrm{e}^{2(\mathrm{i}\theta + \phi)} G^2.$$

If $z_2 = 0$, P is a representative of class N. Otherwise, we can choose $z_2 \exp(\mathrm{i}\theta + \phi)$ to be any nonzero scalar. If we let $z_2 \exp(\mathrm{i}\theta + \phi) = 1$ and

replace $-iz_5 \exp 2(i\theta + \phi)$ by z, we have

$$P = \begin{bmatrix} 0 & 1 & i \\ 1 & z & iz \\ i & iz & -z \end{bmatrix}.$$

We can now summarize the canonical representations of the Petrov matrices. This is done in Table 10.1.

In the next section, it will be shown how the symmetry of principal null directions is related to the six Petrov classes. Investigators using the Newmann–Penrose formalism of spinors have been able to exploit these symmetries to obtain exact solutions to Einstein's field equations. Much of this work is summarized in *Exact Solutions of Einstein's Field Equations* by Kramer, Stephani, MacCallum, and Herlt (1980).

There is no reason to believe that similar results could not be achieved

Table 10.1

Class O	$P = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$	
Class I	$P = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$	$\lambda_1 + \lambda_2 + \lambda_3 = 0$ and $\lambda_j \neq \lambda_k$ if $j \neq k$
Class D	$P = \begin{bmatrix} -2\lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$	$\lambda \neq 0$
Class II	$P = \begin{bmatrix} -2\lambda & 0 & 0 \\ 0 & \lambda + 1 & i \\ 0 & i & \lambda - 1 \end{bmatrix}$	$\lambda \neq 0$
Class N	$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & i \\ 0 & i & -1 \end{bmatrix}$	$\lambda = 0$
Class III	$P = \begin{bmatrix} 0 & 1 & i \\ 1 & z & iz \\ i & iz & -z \end{bmatrix}$	$\lambda = 0$

using other formalisms such as Clifford algebra. To demonstrate this, let us consider an almost trivial example.

If we desire a spherically symmetric solution, the principal null directions would have to be radial. In the next section, it will be shown that most of the Petrov classes do not have the high degree of symmetry required for a spherically symmetric metric. The canonical form of a class D Petrov matrix has two principal directions $(\hat{\gamma}_0 \pm \hat{\gamma}_1)$. These two principal null directions can be interpreted as geodesics for incoming and outgoing photons if $\hat{\gamma}_1$ is interpreted as a unit vector in the radial direction.

The canonical form of the Petrov matrix for class D is

$$P = \begin{bmatrix} -2\lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}.$$

If $\lambda = \alpha + i\beta$, then

$$M = \begin{bmatrix} -2\alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha \end{bmatrix} \quad \text{and} \quad N = \begin{bmatrix} -2\beta & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \beta \end{bmatrix}.$$

Matching this up with Eqs. (10.7) and (10.16), we can write down the corresponding curvature 2-forms. We note that

$$\mathcal{R}_{ij} = \tfrac{1}{2} R_{ijkm} \hat{\gamma}^{km}$$

and thus

$$\mathcal{R}_{23} = \mathcal{R}^{23} = -2\alpha\hat{\gamma}^{23} - 2\beta\hat{\gamma}^{10}, \tag{10.27}$$

$$\mathcal{R}_{31} = \mathcal{R}^{31} = \alpha\hat{\gamma}^{31} + \beta\hat{\gamma}^{20}, \tag{10.28}$$

$$\mathcal{R}_{12} = \mathcal{R}^{12} = \alpha\hat{\gamma}^{12} + \beta\hat{\gamma}^{30}, \tag{10.29}$$

$$\mathcal{R}_{10} = -\mathcal{R}^{10} = -2\beta\hat{\gamma}^{23} + 2\alpha\hat{\gamma}^{10}, \tag{10.30}$$

$$\mathcal{R}_{20} = -\mathcal{R}^{20} = \beta\hat{\gamma}^{31} - \alpha\hat{\gamma}^{20}, \tag{10.31}$$

$$\mathcal{R}_{30} = -\mathcal{R}^{30} = \beta\hat{\gamma}^{12} - \alpha\hat{\gamma}^{30}. \tag{10.32}$$

If we hypothesize the same form for the Schwarzschild line element as was done in Chapter 6, we have

$$(\mathrm{d}s)^2 = c^2 f(r)(\mathrm{d}t)^2 - h(r)(\mathrm{d}r)^2 - r^2(\mathrm{d}\theta)^2 - r^2 \sin^2\theta(\mathrm{d}\phi)^2. \tag{10.33}$$

As noted above, we are required to identify $\hat{\gamma}_1$ as the unit vector in the radial direction. Making this identification and using Eq. (10.33), we can now write

$$\begin{aligned} \gamma_t &= c(f(r))^{\frac{1}{2}}\hat{\gamma}_0, \\ \gamma_r &= (h(r))^{\frac{1}{2}}\hat{\gamma}_1, \\ \gamma_\theta &= r\hat{\gamma}_2, \\ \gamma_\phi &= r\sin\theta\hat{\gamma}_3. \end{aligned}$$

The coefficients of $\hat{\gamma}_k$ in these last four equations can be used to transform the curvature 2-forms from a noncoordinate frame to a coordinate frame. From Eqs. (10.27), (10.28), and (10.31) we have

$$\mathscr{R}_{\theta\phi} = r^2 \sin\theta\mathscr{R}_{23} = r^2\sin\theta(-2\alpha\hat{\gamma}^{23} - 2\beta\hat{\gamma}^{10}), \tag{10.34}$$

$$\mathscr{R}_{\phi r} = rh^{\frac{1}{2}}\sin\theta\mathscr{R}_{31} = rh^{\frac{1}{2}}\sin\theta(\alpha\hat{\gamma}^{31} + \beta\hat{\gamma}^{20}), \tag{10.35}$$

$$\mathscr{R}_{\theta t} = rf^{\frac{1}{2}}\mathscr{R}_{20} = crf^{\frac{1}{2}}(\beta\hat{\gamma}^{31} - \alpha\hat{\gamma}^{20}). \tag{10.36}$$

From Eqs. (6.9)–(6.12),

$$\mathbf{\Gamma}_t = \frac{cf'}{4(fh)^{\frac{1}{2}}}\hat{\gamma}^{10},$$

$$\mathbf{\Gamma}_r = 0,$$

$$\mathbf{\Gamma}_\theta = -\frac{1}{2h^{\frac{1}{2}}}\hat{\gamma}^{12},$$

$$\mathbf{\Gamma}_\phi = -\frac{\sin\theta}{2h^{\frac{1}{2}}}\hat{\gamma}^{13} - \frac{\cos\theta}{2}\hat{\gamma}^{23}.$$

Since

$$\tfrac{1}{2}\mathscr{R}_{jk} = \frac{\partial}{\partial u^j}\mathbf{\Gamma}_k - \frac{\partial}{\partial u^k}\mathbf{\Gamma}_j - \mathbf{\Gamma}_j\mathbf{\Gamma}_k + \mathbf{\Gamma}_k\mathbf{\Gamma}_j,$$

we find that

$$\mathscr{R}_{\theta\phi} = \sin\theta\left[\frac{h-1}{h}\right]\hat{\gamma}^{23}, \tag{10.37}$$

$$\mathscr{R}_{\phi r} = \tfrac{1}{2}h^{-(3/2)}h'\sin\theta\hat{\gamma}^{31}, \tag{10.38}$$

$$\mathscr{R}_{\theta t} = \frac{c}{2h} \frac{f'}{f^{\frac{1}{2}}} \hat{\gamma}^{20}. \tag{10.39}$$

Matching these equations up with Eqs. (10.34)–(10.36), we note that $\beta = 0$ and

$$\frac{h-1}{h} = -2\alpha r^2, \tag{10.40}$$

$$h^{-(3/2)} h' = 2\alpha r h^{\frac{1}{2}}, \tag{10.41}$$

$$f' = -2\alpha r h f. \tag{10.42}$$

Equations (10.40)–(10.42) can easily be solved for h, α, and f with the standard result. (See Problem 10.2.) In principle one should check that this solution obtained from three of the six curvature 2-forms is consistent with the equations corresponding to the remaining curvature 2-forms but this is a straightforward task.

Perhaps we have applied a sledge hammer to crush a gnat. However, it is plausible that this same sledge hammer can be used to solve more difficult problems. As noted before, when the Weyl tensor is replaced by the Riemann tensor, Eq. (10.16) becomes Einstein's field equations for empty space. Using this approach we have avoided the task of extracting formulas for the components of the Ricci tensor from the curvature 2-forms.

In Section 10.3, we will show another way that the Petrov classification scheme can be used to obtain the Kerr metric.

Problem 10.1. Suppose $\boldsymbol{v} = (0, 1, \mathrm{i})$. Show that any 3-dimensional, complex-valued null vector which is orthogonal to $\boldsymbol{v}$ must be a scalar multiple of $\boldsymbol{v}$.

Problem 10.2. Eliminate the function α from Eqs. (10.40) and (10.41) to obtain a differential equation for h. Solve the resulting differential equation for h and use the result to obtain the formula for α. Finally solve Eq. (10.42) for f using the usual long-range boundary conditions. Then check your answer with the result in Chapter 6.

10.2 Principal Null Directions

A null vector $\boldsymbol{n} = n_j \gamma^j$ is said to have a *principal null direction* if it satisfies the equation

$$n_{[k} C_{m]ij[p} n_{q]} n^i n^j = 0. \tag{10.43}$$

From the theory of quartic equations, it is known that Eq. (10.43) has

four solutions (unless $C_{ijkm} = 0$). However, it is possible that some of these directions may coincide. Furthermore, the coincidence pattern of a Weyl tensor is a consequence of its Petrov class.

It is also known if the direction of a principal null vector coincides with another then it satisfies a stronger relation than that represented by Eq. (10.43). In particular

$$C_{kij[p}n_{q]}n^i n^j = 0. \tag{10.44}$$

when $\boldsymbol{n}$ represents two or more coincident directions. Furthermore

$$C_{kij[p}n_{q]}n^j = 0 \tag{10.45}$$

when $\boldsymbol{n}$ represents three or more coincident directions. And finally

$$C_{kijp}n^j = 0 \tag{10.46}$$

when $\boldsymbol{n}$ represents the direction of all four principal null directions.

Ann Stehney (1976) demonstrated that the coincidence pattern is related to an algebraic relation between the 2-dimensional planes that pass through these null vectors.

Of particular importance are null planes. A 2-dimensional plane is said to be a *null plane* if it contains one and only one null vector. Such a plane may be characterized by the fact that it is tangent to the light cone. Since a plane can be represented by the vectors that span it, a null plane N that contains the null vector $\boldsymbol{n}$ can be represented in the form

$$N = \boldsymbol{n} \wedge \boldsymbol{u} = \boldsymbol{nu} \tag{10.47}$$

where $\boldsymbol{u}$ is a 1-vector orthogonal to $\boldsymbol{n}$.

Actually $\boldsymbol{u}$ can be replaced by $\boldsymbol{u} + \alpha\boldsymbol{n}$ without changing N. Thus $\boldsymbol{u}$ may be adjusted so that it has no time component. The most general null plane N containing the null vector $\boldsymbol{n}$ may be presented in the form

$$N = \boldsymbol{n} \wedge ((n_p x^p)\boldsymbol{y} - (n_p y^p)\boldsymbol{x})$$

where $\boldsymbol{x}$ and $\boldsymbol{y}$ are arbitrary 1-vectors.

If we define

$$C_{ij}(N) = \tfrac{1}{2}C_{ijkm}N^{km}$$

and

$$C(N) = \tfrac{1}{2}\gamma^{ij}C_{ij}(N) = \tfrac{1}{2}\gamma^{ij}C_{ijkm}N^{km},$$

then

$$\begin{aligned} C_{ij}(N_1) &= \tfrac{1}{2}C_{ijkm}(N_1)^{km} = \tfrac{1}{2}C_{ijkm}n^k n_p(x_1^p y_1^m - y_1^p x_1^m) \\ &= \tfrac{1}{2}(C_{ijkm}n^k n_p - C_{ijkp}n^k n_m)x_1^p y_1^m \\ &= C_{ijk[m}n_{p]}n^k x_1^p y_1^m. \end{aligned}$$

Furthermore

$$\begin{aligned} (N_2C(N_1))_0 &= (\tfrac{1}{2}(N_2)_{pq}\gamma^{pq}\tfrac{1}{4}\gamma^{ij}C_{ijkm}(N_1)^{km})_0 \\ &= \tfrac{1}{8}(N_2)_{pq}(g^{pj}g^{qi} - g^{pi}g^{qj})C_{ijkm}(N_1)^{km} \\ &= -\tfrac{1}{4}(N_2)^{ij}C_{ijkm}(N_1)^{km}. \end{aligned}$$

If $N_2 = \boldsymbol{n} \wedge ((n_q x_2^q)\boldsymbol{y}_2 - (n_q y_2^q)\boldsymbol{x}_2)$, then

$$\begin{aligned} (N_2C(N_1))_0 &= -\tfrac{1}{4}(N_2)^{ij}C_{ijkm}(N_1)^{km} \\ &= n_{[q}C_{j]ik[m}n_{p]}n^i n^k x_1^p y_1^m x_2^q y_2^j. \end{aligned}$$

Since N_1 and N_2 are arbitrary null planes containing $\boldsymbol{n}$, it follows that x_1^p, y_1^m, x_2^q, and y_2^j are arbitrary and thus

$$\begin{aligned} (N_2C(N_1))_0 &= -\tfrac{1}{4}(N_2)^{ij}C_{ijkm}(N_1)^{km} = 0 \\ &\Leftrightarrow n_{[q}C_{j]ik[m}n_{p]}n^i n^k = 0. \end{aligned}$$

Thus we see that the condition that $\boldsymbol{n}$ is a principal null vector is equivalent to the condition that $(N_2)^{ij}C_{ijkm}(N_1)^{km} = 0$ where N_1 and N_2 are arbitrary null planes containing $\boldsymbol{n}$. Actually the condition that $(N_2)^{ij}C_{ijkm}(N_1)^{km} = 0$ can be replaced by the condition that $N^{ij}C_{ijkm}N^{km} = N_{ij}C^{ij}{}_{km}N^{km} = 0$ for all null planes N containing $\boldsymbol{n}$. (See Problem 10.3.) Thus to determine our desired null planes we need to solve two equations:

$$N_{ij}C^{ij}{}_{km}N^{km} = 0, \tag{10.48}$$

and

$$N_{ij}N^{ij} = 0. \tag{10.49}$$

Solutions to these two equations may be constructed from solutions of some very closely related equations involving the Petrov matrix P. Namely

$$\vec{v}^{\mathrm{T}}\vec{w} = 0 \tag{10.50}$$

and

$$\vec{v}^{\mathrm{T}}\vec{v} = 0 \tag{10.51}$$

where

$$\vec{w} = P\vec{v} = (M + \mathrm{i}N)\vec{v}. \tag{10.52}$$

It is understood here that $\vec{v}$ has complex components and if

$$\vec{v} = \begin{bmatrix} v^1 \\ v^2 \\ v^3 \end{bmatrix}$$

then $\vec{v}^{\mathrm{T}} = (v^1, v^2, v^3)$. To construct a solution for N from $\vec{v}$, we note that if $\vec{v}$ and $\vec{w}$ satisfy Eq. (10.52) then

$$\begin{bmatrix} M & N \\ -N & M \end{bmatrix}\begin{bmatrix} \mathrm{e}^{\mathrm{i}\theta}\vec{v} \\ \mathrm{i}\mathrm{e}^{\mathrm{i}\theta}\vec{v} \end{bmatrix} = \begin{bmatrix} \mathrm{e}^{\mathrm{i}\theta}\vec{w} \\ \mathrm{i}\mathrm{e}^{\mathrm{i}\theta}\vec{w} \end{bmatrix} \tag{10.53}$$

where θ is arbitrary. If $\vec{v} = \vec{p} + \mathrm{i}\vec{q}$ and $\vec{w} = \vec{r} + \mathrm{i}\vec{s}$ where $\vec{p}$, $\vec{q}$, $\vec{r}$, and $\vec{s}$ are real, then we can take the real part of Eq. (10.53) to get

$$\begin{bmatrix} M & N \\ -N & M \end{bmatrix}\begin{bmatrix} \vec{p}\cos\theta - \vec{q}\sin\theta \\ -\vec{p}\sin\theta - \vec{q}\cos\theta \end{bmatrix} = \begin{bmatrix} \vec{r}\cos\theta - \vec{s}\sin\theta \\ -\vec{r}\sin\theta - \vec{s}\cos\theta \end{bmatrix}.$$

If $\vec{v}^{\mathrm{T}}\vec{v} = 0$, then $\vec{p}^{\mathrm{T}}\vec{p} - \vec{q}^{\mathrm{T}}\vec{q} + 2\mathrm{i}\vec{p}^{\mathrm{T}}\vec{q} = 0$. Thus $|\vec{p}| = |\vec{q}|$ and $\vec{p}^{\mathrm{T}}\vec{q} = 0$. The reader is now in a position to check the statement that if $\vec{v}$ satisfies Eqs. (10.50)–(10.52) then Eqs. (10.48), (10.49) are satisfied by N if

$$\begin{aligned} N = {} & \hat{\gamma}_{23}(p^1\cos\theta - q^1\sin\theta) + \hat{\gamma}_{31}(p^2\cos\theta - q^2\sin\theta) \\ & + \hat{\gamma}_{12}(p^3\cos\theta - q^3\sin\theta) - \hat{\gamma}_{10}(p^1\sin\theta + q^1\cos\theta) \\ & - \hat{\gamma}_{20}(p^2\sin\theta + q^2\cos\theta) - \hat{\gamma}_{30}(p^3\sin\theta + q^3\cos\theta). \end{aligned} \tag{10.54}$$

To extract the null vector $\boldsymbol{n}$ from the null plane N, it is useful to review the comments following Eq. (10.47). Examining the coefficients of $\hat{\gamma}_{10}$, $\hat{\gamma}_{20}$, and $\hat{\gamma}_{30}$ in Eq. (10.54), we can read off the components of $\boldsymbol{u}$. From Eq. (10.47), it is clear that

$$\boldsymbol{n} = -\boldsymbol{N}\boldsymbol{u} = \hat{\gamma}_0 + \hat{\gamma}_1(q^2p^3 - q^3p^2) + \hat{\gamma}_2(q^3p^1 - q^1p^3) + \hat{\gamma}_3(q^1p^2 - q^2p^1). \tag{10.55}$$

We now have the machinery to compute the principal null directions for the canonical representations of each of the Petrov classes. For class I (see Table 10.1):

$$P\vec{v} = \begin{bmatrix} \lambda_1 v^1 \\ \lambda_2 v^2 \\ \lambda_3 v^3 \end{bmatrix}.$$

Thus $\vec{v}$ must satisfy the equations

$$\vec{v}^{\mathrm{T}} P \vec{v} = (v^1)^1 \lambda_1 + (v^2)^2 \lambda_2 + (v^3)^2 \lambda_3 = 0$$

and

$$\vec{v}^{\mathrm{T}} \vec{v} = (v^1)^2 + (v^2)^2 + (v^3)^2 = 0.$$

From these last two equations, we see that the vector $((v^1)^2, (v^2)^2, (v^3)^2)$ is orthogonal to both $(\lambda_1, \lambda_2, \lambda_3)$ and $(1, 1, 1)$ and is therefore a scalar multiple of $(\lambda_1, \lambda_2, \lambda_3) \times (1, 1, 1)$. Thus

$$(v^1)^2 = r^2 \, \mathrm{e}^{\mathrm{i}2\theta}(\lambda_2 - \lambda_3) = (a^1 + \mathrm{i}b^1)^2, \tag{10.56}$$

$$(v^2)^2 = r^2 \, \mathrm{e}^{\mathrm{i}2\theta}(\lambda_3 - \lambda_1) = (a^2 + \mathrm{i}b^2)^2, \tag{10.57}$$

$$(v^3)^2 = r^2 \, \mathrm{e}^{\mathrm{i}2\theta}(\lambda_1 - \lambda_2) = (a^3 + \mathrm{i}b^3)^2. \tag{10.58}$$

It is understood that r is adjusted so that $\vec{a}^{\mathrm{T}}\vec{a} = \vec{b}^{\mathrm{T}}\vec{b} = 1$. From Eqs. (10.56), (10.57), and (10.58), we can extract four distinct principal null directions by letting

$$\vec{v}_1^{\mathrm{T}} = (a^1 + \mathrm{i}b^1, a^2 + \mathrm{i}b^2, a^3 + \mathrm{i}b^3), \tag{10.59}$$

$$\vec{v}_2^{\mathrm{T}} = (a^1 + \mathrm{i}b^1, a^2 + \mathrm{i}b^2, -a^3 - \mathrm{i}b^3), \tag{10.60}$$

$$\vec{v}_3^{\mathrm{T}} = (a^1 + \mathrm{i}b^1, -a^2 - \mathrm{i}b^2, a^3 + \mathrm{i}b^3), \tag{10.61}$$

$$\vec{v}_4^{\mathrm{T}} = (-a^1 - \mathrm{i}b^1, a^2 + \mathrm{i}b^2, a^3 + \mathrm{i}b^3). \tag{10.62}$$

For the first case

$$\boldsymbol{n}_1 = \hat{\gamma}_0 + \hat{\gamma}_1(b^2a^3 - b^3a^2) + \hat{\gamma}_2(b^3a^1 - b^1a^3) + \hat{\gamma}_3(b^1a^2 - b^2a^1). \tag{10.63}$$

For the canonical representation of Petrov's class I, there is a substantial degree of symmetry. To obtain $\boldsymbol{n}_2$ from $\boldsymbol{n}_1$, we change the signs of a^3 and b^3. This has the effect of reversing the signs for the components of $\hat{\gamma}_1$ and $\hat{\gamma}_2$ in $\boldsymbol{n}_1$. This corresponds to a 180° rotation in the 1–2 plane. Similarly $\boldsymbol{n}_3$ can be obtained from $\boldsymbol{n}_1$ by a 180° rotation in the 1–3 plane. Finally $\boldsymbol{n}_4$ can be obtained from $\boldsymbol{n}_1$ by a 180° rotation in the 2–3 plane. The three space angles between these directions would be preserved under spatial rotations but not under boosts.

Now let us consider class D. In some sense class D is a special case of class I. For class D, $\lambda_2 = \lambda_3 = \lambda$ and $\lambda_1 = -2\lambda$. For Eq. (10.56), $a^1 = b^1 = 0$. From Eqs. (10.59)–(10.62), it is clear that there are two principal null directions: $\hat{\gamma}_0 + \hat{\gamma}_1$ and $\hat{\gamma}_0 - \hat{\gamma}_1$. A further examination of the same equations

reveals that one of these directions results from the coincidence of $\boldsymbol{n}_1$ and $\boldsymbol{n}_4$, while the other results from the coincidence of $\boldsymbol{n}_2$ and $\boldsymbol{n}_3$.

For class II, there is one doublet principal null direction and two singlet principal null directions. For the canonical representation of class II, it is useful to let $\lambda = (a + ib)^2$. Calculation shows that the doublet direction is $\hat{\gamma}_0 - \hat{\gamma}_1$ and the two singlet directions are

$$\hat{\gamma}_0 + \frac{3(a^2 + b^2) - 1}{3(a^2 + b^2) + 1}\hat{\gamma}_1 - \frac{2\sqrt{3}a}{3(a^2 + b^2) + 1}\hat{\gamma}_2 - \frac{2\sqrt{3}b}{3(a^2 + b^2) + 1}\hat{\gamma}_3$$

and

$$\hat{\gamma}_0 + \frac{3(a^2 + b^2) - 1}{3(a^2 + b^2) + 1}\hat{\gamma}_1 + \frac{2\sqrt{3}a}{3(a^2 + b^2) + 1}\hat{\gamma}_2 + \frac{2\sqrt{3}b}{3(a^2 + b^2) + 1}\hat{\gamma}_3.$$

We note that one singlet direction can be obtained from the other by a 180° rotation in the 2–3 plane.

Class N is obtained from class II by setting $\lambda = a = b = 0$. In that case there is a single null direction which is $\hat{\gamma}_0 - \hat{\gamma}_1$.

For class III, there is one triplet principal null direction. For the canonical form that we have chosen, this principal null direction is $\hat{\gamma}_0 - \hat{\gamma}_1$. If $\lambda = a + ib$, the singlet direction is

$$\hat{\gamma}_0 + \frac{4 - a^2 - b^2}{4 + a^2 + b^2}\hat{\gamma}_1 + \frac{4a}{4 + a^2 + b^2}\hat{\gamma}_2 - \frac{4b}{4 + a^2 + b^2}\hat{\gamma}_3.$$

Problem 10.3. Show that the condition that $(N_2)^{ij}C_{ijkm}(N_1)^{km} = 0$ for arbitrary null planes N_1 and N_2 passing through the null vector n is equivalent to the condition that $N^{ij}C_{ijkm}N^{km} = 0$ for any null plane N passing through $\boldsymbol{n}$. Hint: suppose N_1 and N_2 represent distinct null planes passing through $\boldsymbol{n}$. Show that any null plane N passing through $\boldsymbol{n}$ may be represented in the form $N = aN_1 + bN_2$ where a and b are real scalars. One may also use the fact that $(N_2)^{ij}C_{ijkm}(N_1)^{km} = (N_1)^{ij}C_{ijkm}(N_2)^{km}$.

Problem 10.4. In Section 5.5, it was pointed out that a vector that is parallel transported around a small closed loop undergoes an infinitesimal rotation. The rotation operator that achieves this is $I + \frac{1}{2}(\Delta x^i)(\Delta y^j)\mathscr{R}_{ij}$ where the loop lies in the plane $\boldsymbol{x} \wedge \boldsymbol{y}$. A vector that undergoes a rotation does not change its length. However, a null vector with its zero length may be rotated in such a fashion that its direction remains unchanged but it is multiplied by some scalar. The condition that a null vector $\boldsymbol{n}$ does not change direction when it is parallel transported about the loop $\Delta\boldsymbol{x} \wedge \Delta\boldsymbol{y}$ is $[\frac{1}{2}(\Delta x^i\, \Delta y^j)\mathscr{R}_{ij}, \boldsymbol{n}] = \alpha\boldsymbol{n}$ or

$$[\tfrac{1}{2}(\Delta x^i\, \Delta y^j)\mathscr{R}_{ij}, \boldsymbol{n}] \wedge \boldsymbol{n} = 0. \tag{10.64}$$

1. Show Eq. (10.64) is equivalent to Eq. (10.43) if the loop lies in a null plane N containing the principal null vector $\boldsymbol{n}$.
2. Show that Eq. (10.44) is equivalent to the condition that if the null vector $\boldsymbol{n}$ is parallel transported about a small closed loop in any null plane N containing $\boldsymbol{n}$, then $\boldsymbol{n}$ will be completely unchanged.
3. Show that the condition that $\boldsymbol{n}$ corresponds to the coincidence of two principal null directions is equivalent to the condition that $C(N) = \alpha N$ where N is an arbitrary null plane containing N.
4. Show that the condition for a triple coincidence (Eq. (10.45)) is equivalent to the condition that $C(N) = 0$.
5. Show that the condition that all four principal null directions coincide is equivalent to the condition that $C(A) = 0$ where A is an arbitrary (not necessarily null) plane containing the principal null vector $\boldsymbol{n}$.
6. Determine the consequence of parallel transport of various vectors in various planes when there is a triple or quadruple coincidence of the principal null directions.

10.3 The Kerr Metric Revisited via its Petrov Matrix

The Kerr metric was first computed by the use of spinors. An advantage of using spinors is that one can introduce a principal null direction into the computations in a natural way. However, one can also deal with principal null directions using Clifford algebra without difficulty.

In this section, we will demonstrate how the Petrov classification scheme can be used to obtain the Kerr metric in a somewhat more straightforward manner than was achieved in Chapter 9.

As in Chapter 9, we note that the Kerr metric is characterized by the equation:

$$\mathrm{d}\boldsymbol{s} = \gamma_\alpha \,\mathrm{d}x^\alpha$$

where

$$\gamma_k = \hat{\gamma}_{\underline{k}} - m w_{\underline{k}} w^{\underline{p}} \hat{\gamma}_{\underline{p}}. \tag{10.65}$$

The γ_k's form the basis of a coordinate frame while the $\hat{\gamma}_{\underline{k}}$'s with the underlined indices form the basis of an orthonormal noncoordinate frame. In particular

$$\hat{\gamma}_{\underline{j}}\hat{\gamma}_{\underline{k}} + \hat{\gamma}_{\underline{k}}\hat{\gamma}_{\underline{j}} = 2n_{\underline{jk}}$$

where

$$n_{\underline{00}} = 1, \qquad n_{\underline{11}} = n_{\underline{22}} = n_{\underline{33}} = -1, \qquad \text{and} \qquad n_{\underline{jk}} = 0 \text{ for } \underline{j} \neq \underline{k}.$$

As in Chapter 9, $\boldsymbol{w}$ is a null vector. Since

$$w = w^k\gamma_k = w^{\underline{k}}\hat{\gamma}_{\underline{k}} = w^{\underline{k}}\gamma_k + mw^{\underline{k}}w_{\underline{k}}w^{\underline{p}}\hat{\gamma}_{\underline{p}} = w^{\underline{k}}\gamma_k,$$

it follows that $w^k = w^{\underline{k}}$. Thus it is unnecessary to underline the indices for the components of the null vector $\boldsymbol{w}$.

The fact that one might find a solution to Einstein's field equations with the form of the line element imposed by Eq. (10.65) is justified by the fact that Eddington's form of the Schwarzschild metric has the same form.

As in Chapter 9, the Fock–Ivanenko 2-vectors are

$$\boldsymbol{\Gamma}_\alpha = -\frac{m}{2}\gamma^{pq}\frac{\partial}{\partial x^p}(w_\alpha w_q). \tag{10.66}$$

Also

$$\tfrac{1}{2}\mathcal{R}_{jk} = \frac{\partial}{\partial x^j}\boldsymbol{\Gamma}_k - \frac{\partial}{\partial x^k}\boldsymbol{\Gamma}_j - \boldsymbol{\Gamma}_j\boldsymbol{\Gamma}_k + \boldsymbol{\Gamma}_k\boldsymbol{\Gamma}_j, \tag{10.67}$$

where

$$\mathcal{R}_{jk} = \tfrac{1}{2}R_{jkmn}\gamma^{mn}. \tag{10.68}$$

In view of Eqs. (10.65)–(10.67), it is clear that $\mathcal{R}_{jk}$ is a polynomial in the variable m with the powers of m ranging from one up to possibly four. In this section, we will deal with only the first-order term.

In the subsequent computations, we will assume that the null vector $w_p\hat{\gamma}^{\underline{p}}$ that appears in Eq. (10.65) is a principal null vector. As in Chapter 9, it is useful to introduce a 3-dimensional space vector $\vec{\lambda}$ of unit length. In particular

$$w_p\gamma^{\underline{p}} = w_0[\hat{\gamma}^{\underline{0}} + \hat{\gamma}^{\underline{1}}\lambda_{\underline{1}} + \hat{\gamma}^{\underline{2}}\lambda_{\underline{2}} + \hat{\gamma}^{\underline{3}}\lambda_{\underline{3}}]. \tag{10.69}$$

The canonical form for the class D Petrov matrix is

$$P = \begin{bmatrix} -2\nu & 0 & 0 \\ 0 & \nu & 0 \\ 0 & 0 & \nu \end{bmatrix}. \tag{10.70}$$

In the previous section, it was shown that the principal null vectors for this canonical form are $\hat{\gamma}^{\underline{0}} \pm \hat{\gamma}^{\underline{1}}$. For our purpose, we need to obtain the form of the Petrov matrix when one of the principal null vectors is $\hat{\gamma}^{\underline{0}} + \hat{\gamma}^{\underline{k}}\lambda_{\underline{k}}$. Our first thought is to consider what happens to the Petrov matrix when the vector $\hat{\gamma}^{\underline{1}}$ is rotated into the vector $\hat{\gamma}^{\underline{k}}\lambda_{\underline{k}}$.

If double primes are used to designate the frame for the Petrov canonical

form and single primes are used to designate the rotated frame then

$$\overset{\prime\prime}{\gamma}{}^{\underline{1}} = \overset{\prime}{\gamma}{}^{\underline{k}}\lambda_{\underline{k}} = \mathscr{R}\overset{\prime}{\gamma}{}^{\underline{1}}\mathscr{R}^{-1}, \tag{10.71}$$

where $\mathscr{R}$ is a rotation operator. There are many rotation operators that will achieve this but all will have the same effect on the Petrov matrix. One rotation that will achieve our purpose is the 180° rotation about the axis that is half-way between $(1, 0, 0)$ and $(\lambda_1, \lambda_2, \lambda_3)$. To write down our desired rotation operator, we need to normalize the vector representing the axis of rotation which is $(1 + \lambda_1, \lambda_2, \lambda_3)$. We see that $(1 + \lambda_1)^2 + (\lambda_2)^2 + (\lambda_3)^2 = (1 + \lambda_1)^2 + 1 - (\lambda_1)^2 = 2(1 + \lambda_1)$. Therefore, our rotation operator is

$$\begin{aligned}
\mathscr{R} &= I\cos\frac{\theta}{2} + \sin\frac{\theta}{2}\,[n^1\overset{\prime}{\gamma}{}^{\underline{23}} + n^2\overset{\prime}{\gamma}{}^{\underline{31}} + n^3\overset{\prime}{\gamma}{}^{\underline{12}}] \\
&= \frac{1}{\sqrt{2(1+\lambda_1)}}\,[(1+\lambda_1)\overset{\prime}{\gamma}{}^{\underline{23}} + \lambda_2\overset{\prime}{\gamma}{}^{\underline{31}} + \lambda_3\overset{\prime}{\gamma}{}^{\underline{12}}].
\end{aligned}$$

Using this rotation operator,

$$\begin{aligned}
\overset{\prime\prime}{\gamma}{}^{\underline{23}} &= \mathscr{R}\overset{\prime}{\gamma}{}^{\underline{23}}\mathscr{R}^{-1} \\
&= \frac{1}{2(1+\lambda_1)}\,\overset{\prime}{\gamma}{}^{\underline{23}}[(1+\lambda_1)\overset{\prime}{\gamma}{}^{\underline{23}} - \lambda_2\overset{\prime}{\gamma}{}^{\underline{31}} - \lambda_3\overset{\prime}{\gamma}{}^{\underline{12}}] \\
&\quad \times [(1+\lambda_1)\overset{\prime}{\gamma}{}^{\underline{23}} + \lambda_2\overset{\prime}{\gamma}{}^{\underline{31}} + \lambda_3\overset{\prime}{\gamma}{}^{\underline{12}}] \\
&= -\overset{\prime}{\gamma}{}^{\underline{23}}[-I\lambda_1 - \overset{\prime}{\gamma}{}^{\underline{31}}\lambda_3 + \overset{\prime}{\gamma}{}^{\underline{12}}\lambda_2],
\end{aligned}$$

or

$$\overset{\prime\prime}{\gamma}{}^{\underline{23}} = \overset{\prime}{\gamma}{}^{\underline{23}}\lambda_1 + \overset{\prime}{\gamma}{}^{\underline{31}}\lambda_2 + \overset{\prime}{\gamma}{}^{\underline{12}}\lambda_3. \tag{10.72}$$

Similarly

$$\overset{\prime\prime}{\gamma}{}^{\underline{31}} = \overset{\prime}{\gamma}{}^{\underline{23}}\lambda_2 + \overset{\prime}{\gamma}{}^{\underline{31}}\left(-1 + \frac{(\lambda_2)^2}{1+\lambda_1}\right) + \overset{\prime}{\gamma}{}^{\underline{12}}\frac{\lambda_2\lambda_3}{1+\lambda_1}, \tag{10.73}$$

and

$$\overset{\prime\prime}{\gamma}{}^{\underline{12}} = \overset{\prime}{\gamma}{}^{\underline{23}}\lambda_3 + \overset{\prime}{\gamma}{}^{\underline{31}}\frac{\lambda_2\lambda_3}{1+\lambda_1} + \overset{\prime}{\gamma}{}^{\underline{12}}\left(-1 + \frac{(\lambda_2)^2}{1+\lambda_1}\right). \tag{10.74}$$

From Eq. (10.18), we see that if P'' is a Petrov matrix, then

$$P''\begin{bmatrix} \overset{\prime\prime}{a}{}^{\underline{23}} - i\overset{\prime\prime}{a}{}^{\underline{10}} \\ \overset{\prime\prime}{a}{}^{\underline{31}} - i\overset{\prime\prime}{a}{}^{\underline{20}} \\ \overset{\prime\prime}{a}{}^{\underline{12}} - i\overset{\prime\prime}{a}{}^{\underline{30}} \end{bmatrix} = \begin{bmatrix} \overset{\prime\prime}{b}{}^{\underline{23}} - i\overset{\prime\prime}{b}{}^{\underline{10}} \\ \overset{\prime\prime}{b}{}^{\underline{31}} - i\overset{\prime\prime}{b}{}^{\underline{20}} \\ \overset{\prime\prime}{b}{}^{\underline{12}} - i\overset{\prime\prime}{b}{}^{\underline{30}} \end{bmatrix}. \tag{10.75}$$

However, the entities $\overset{\prime\prime}{a}{}^{\underline{ij}} - i\overset{\prime\prime}{a}{}^{\underline{k0}}$ transform in the same way as $\overset{\prime\prime}{\gamma}{}^{\underline{ij}} - i\overset{\prime\prime}{\gamma}{}^{\underline{k0}} =$

$(I + iJ)\overset{\prime\prime}{\hat{\gamma}}{}^{\underline{ij}}$. Thus if we multiply Eqs. (10.72–(10.74) by $(I + iJ)$ and read off the appropriate coefficients, we see that

$$\begin{bmatrix} \overset{\prime\prime}{a}{}^{\underline{23}} - i\overset{\prime\prime}{a}{}^{\underline{10}} \\ \overset{\prime\prime}{a}{}^{\underline{31}} - i\overset{\prime\prime}{a}{}^{\underline{20}} \\ \overset{\prime\prime}{a}{}^{\underline{12}} - i\overset{\prime\prime}{a}{}^{\underline{30}} \end{bmatrix} = R \begin{bmatrix} \overset{\prime}{a}{}^{\underline{23}} - i\overset{\prime}{a}{}^{\underline{10}} \\ \overset{\prime}{a}{}^{\underline{31}} - i\overset{\prime}{a}{}^{\underline{20}} \\ \overset{\prime}{a}{}^{\underline{12}} - i\overset{\prime}{a}{}^{\underline{30}} \end{bmatrix} \tag{10.76}$$

where

$$R = \begin{bmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_2 & -1 + \dfrac{(\lambda_2)^2}{1 + \lambda_1} & \dfrac{\lambda_2 \lambda_3}{1 + \lambda_1} \\ \lambda_3 & \dfrac{\lambda_2 \lambda_3}{1 + \lambda_1} & -1 + \dfrac{(\lambda_3)^2}{1 + \lambda_1} \end{bmatrix}. \tag{10.77}$$

From Eqs. (10.75) and (10.76), we have

$$P''R \begin{bmatrix} \overset{\prime}{a}{}^{\underline{23}} - i\overset{\prime}{a}{}^{\underline{10}} \\ \overset{\prime}{a}{}^{\underline{31}} - i\overset{\prime}{a}{}^{\underline{20}} \\ \overset{\prime}{a}{}^{\underline{12}} - i\overset{\prime}{a}{}^{\underline{30}} \end{bmatrix} = R \begin{bmatrix} \overset{\prime}{b}{}^{\underline{23}} - i\overset{\prime}{b}{}^{\underline{10}} \\ \overset{\prime}{b}{}^{\underline{31}} - i\overset{\prime}{b}{}^{\underline{20}} \\ \overset{\prime}{b}{}^{\underline{12}} - i\overset{\prime}{b}{}^{\underline{30}} \end{bmatrix}$$

$$P' = R^{-1}P''R = R^{-1} \begin{bmatrix} -2v & 0 & 0 \\ 0 & v & 0 \\ 0 & 0 & v \end{bmatrix} R.$$

If we note that R^{-1} is the transpose of R and carry out the matrix multiplication, we get

$$P' = v \begin{bmatrix} 1 - 3(\lambda_1)^2 & -3\lambda_1\lambda_2 & -3\lambda_1\lambda_3 \\ -3\lambda_2\lambda_1 & 1 - 3(\lambda_2)^2 & -3\lambda_2\lambda_3 \\ -3\lambda_3\lambda_1 & -3\lambda_3\lambda_2 & 1 - 3(\lambda_3)^2 \end{bmatrix}. \tag{10.78}$$

After all that computation, it turns out that P' is *not* the Petrov matrix for the Kerr metric. As was noted at the beginning of the calculation, for the canonical form of Petrov's matrix, the principal null directions are $\hat{\gamma}^{\underline{0}} \pm \hat{\gamma}^{\underline{1}}$. The rotation has carried these two vectors into $\hat{\gamma}^{\underline{0}} \pm \hat{\gamma}^{\underline{k}}\lambda_k$. Thus the spatial components of the two null vectors are still aligned. This alignment does exist for the highly symmetric Schwarzschild metric where the principal null directions correspond to that of ingoing and outgoing radial directions. However, this alignment does not exist for the Kerr metric.

For that reason, we must consider an additional Lorentz transformation

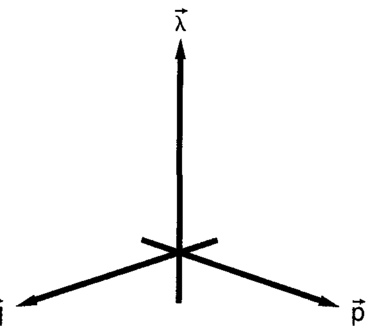


Fig. 10.1. The 3-dimensional unit space vectors $\vec{\lambda}$, $\vec{p}$, and $\vec{q}$ are mutually perpendicular with the relative orientation shown.

that will preserve the null vector $\gamma^{\underline{0}} + \gamma^{\underline{k}}\lambda_{\underline{k}}$ but not the null vector $\gamma^{\underline{0}} - \gamma^{\underline{k}}\lambda_{\underline{k}}$. Two Lorentz transformations which will not work are a spatial rotation about the axis $\vec{\lambda}$ or a pure boost in the direction of $\vec{\lambda}$. A Lorentz transformation that does achieve the desired effect is a null rotation, namely

$$\mathscr{L} = \exp\left(-\frac{\phi}{2}\hat{p} \wedge (\gamma^{\underline{0}} + \gamma^{\underline{k}}\lambda_{\underline{k}})\right), \tag{10.79}$$

where

$$\hat{p} = \gamma^{\underline{j}} p_{\underline{j}},\ p_{\underline{0}} = 0,\ (\hat{p})^2 = -1, \text{ and } \hat{p} \text{ is orthogonal to } \hat{\lambda} = \gamma^{\underline{k}}\lambda_{\underline{k}}.$$

From Eq. (10.79), we can write

$$\mathscr{L} = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{-\phi}{2}\right)^n (\hat{p}(\gamma^{\underline{0}} + \gamma^{\underline{k}}\lambda_{\underline{k}}))^n,$$

but

$$[\hat{p}(\gamma^{\underline{0}} + \gamma^{\underline{k}}\lambda_{\underline{k}})]^2 = 0,$$

so

$$\mathscr{L} = I - \frac{\phi}{2}\hat{p}(\gamma^{\underline{0}} + \gamma^{\underline{k}}\lambda_{\underline{k}})$$

or

$$\mathscr{L} = I - \frac{\phi}{2}[\gamma^{\underline{k0}} p_{\underline{k}} + \gamma^{\underline{23}} q_{\underline{1}} + \gamma^{\underline{31}} q_{\underline{2}} + \gamma^{\underline{12}} q_{\underline{3}}] \tag{10.80}$$

where $(q_{\underline{1}}, q_{\underline{2}}, q_{\underline{3}}) = (p_{\underline{1}}, p_{\underline{2}}, p_{\underline{3}}) \times (\lambda_{\underline{1}}, \lambda_{\underline{2}}, \lambda_{\underline{3}})$. We see that $\vec{\lambda}$, $\vec{p}$, and $\vec{q}$ are unit space vectors which are mutually perpendicular with the orientation shown in Fig. 10.1.

Using the null rotation operator $\mathscr{L}$, we have

$$\begin{aligned}(I + \mathrm{i}J)\hat{\gamma}'^{\underline{23}} &= \hat{\gamma}'^{\underline{23}} - \mathrm{i}\hat{\gamma}'^{\underline{10}} = \mathscr{L}(\hat{\gamma}^{\underline{23}} - \mathrm{i}\hat{\gamma}^{\underline{10}})\mathscr{L}^{-1} \\ &= \mathscr{L}(I + \mathrm{i}J)\hat{\gamma}^{\underline{23}}\mathscr{L}^{-1} = (I + \mathrm{i}J)\mathscr{L}\hat{\gamma}^{\underline{23}}\mathscr{L}^{-1}.\end{aligned}$$

It is useful to note that

$$(I + \mathrm{i}J)\hat{\gamma}^{\underline{10}} = (I + \mathrm{i}J)(\mathrm{i}J\hat{\gamma}^{\underline{10}}) = (I + \mathrm{i}J)\mathrm{i}\hat{\gamma}^{\underline{23}}.$$

In a similar fashion

$$\begin{aligned}(I+\mathrm{i}J)\mathscr{L} &= (I+\mathrm{i}J)\left(I - \frac{\phi}{2}\left[\hat{\gamma}^{\underline{23}}(q_{\underline{1}} + \mathrm{i}p_{\underline{1}}) + \hat{\gamma}^{\underline{31}}(q_{\underline{2}} + \mathrm{i}p_{\underline{2}}) + \hat{\gamma}^{\underline{12}}(q_{\underline{3}} + \mathrm{i}p_{\underline{3}})\right]\right) \\ &= (I + \mathrm{i}J)\left[I - \frac{\phi}{2}(\hat{\gamma}^{\underline{23}}z_{\underline{1}} + \hat{\gamma}^{\underline{31}}z_{\underline{2}} + \hat{\gamma}^{\underline{12}}z_{\underline{3}})\right] \\ &= (I + \mathrm{i}J)\bar{\mathscr{L}}\end{aligned}$$

where

$$\bar{\mathscr{L}} \equiv I - \frac{\phi}{2}(\hat{\gamma}^{\underline{23}}z_{\underline{1}} + \hat{\gamma}^{\underline{31}}z_{\underline{2}} + \hat{\gamma}^{\underline{12}}z_{\underline{3}}) \tag{10.81}$$

and

$$z_{\underline{k}} = q_{\underline{k}} + \mathrm{i}p_{\underline{k}}. \tag{10.82}$$

From virtually identical manipulations, we also have

$$(I + \mathrm{i}J)\mathscr{L}^{-1} = (I + \mathrm{i}J)\bar{\mathscr{L}}^{-1}.$$

Since $\frac{1}{2}(I + \mathrm{i}J)(I + \mathrm{i}J) = (I + \mathrm{i}J)$, we now have

$$\begin{aligned}(I + \mathrm{i}J)\mathscr{L}\hat{\gamma}^{\underline{23}}\mathscr{L}^{-1} &= \tfrac{1}{2}(I + \mathrm{i}J)\mathscr{L}\hat{\gamma}^{\underline{23}}(I + \mathrm{i}J)\mathscr{L}^{-1} \\ &= \tfrac{1}{2}(I + \mathrm{i}J)\bar{\mathscr{L}}\hat{\gamma}^{\underline{23}}(I + \mathrm{i}J)\bar{\mathscr{L}}^{-1} \\ &= (I + \mathrm{i}J)\bar{\mathscr{L}}\hat{\gamma}^{\underline{23}}\bar{\mathscr{L}}^{-1}.\end{aligned}$$

With these results, we can write

$$\begin{aligned}(I + \mathrm{i}J)\hat{\gamma}'^{\underline{23}} &= (I + \mathrm{i}J)\left[I - \frac{\phi}{2}(\hat{\gamma}^{\underline{23}}z_{\underline{1}} + \hat{\gamma}^{\underline{31}}z_{\underline{2}} + \hat{\gamma}^{\underline{12}}z_{\underline{3}})\right] \\ &\quad\times \hat{\gamma}^{\underline{23}}\left[I + \frac{\phi}{2}(\hat{\gamma}^{\underline{23}}z_{\underline{1}} + \hat{\gamma}^{\underline{31}}z_{\underline{2}} + \hat{\gamma}^{\underline{12}}z_{\underline{3}})\right] \\ &= (I + \mathrm{i}J)\hat{\gamma}^{\underline{23}}\left[I - \frac{\phi}{2}(\hat{\gamma}^{\underline{23}}z_{\underline{1}} - \hat{\gamma}^{\underline{31}}z_{\underline{2}} - \hat{\gamma}^{\underline{12}}z_{\underline{3}})\right] \\ &\quad\times \left[I + \frac{\phi}{2}(\hat{\gamma}^{\underline{23}}z_{\underline{1}} + \hat{\gamma}^{\underline{31}}z_{\underline{2}} + \hat{\gamma}^{\underline{12}}z_{\underline{3}})\right].\end{aligned}$$

To complete this computation, it is useful to note that from Eq. (10.82), it follows that

$$
\begin{aligned}
(z_{\underline{1}})^2 + (z_{\underline{2}})^2 + (z_{\underline{3}})^2 &= \sum_{\underline{k}=\underline{1}}^{\underline{3}} (p_{\underline{k}} p_{\underline{k}} + 2\mathrm{i} p_{\underline{k}} q_{\underline{k}} - q_{\underline{k}} q_{\underline{k}}) \\
&= 1 + 0 - 1 = 0.
\end{aligned}
$$

Completing the calculation for $(I + \mathrm{i}J)\acute{\gamma}^{\underline{23}}$, and then computing both $(I + \mathrm{i}J)\acute{\gamma}^{\underline{31}}$ and $(I + \mathrm{i}J)\acute{\gamma}^{\underline{12}}$, we find that

$$
\begin{bmatrix} (I + \mathrm{i}J)\acute{\gamma}^{\underline{23}} \\ (I + \mathrm{i}J)\acute{\gamma}^{\underline{31}} \\ (I + \mathrm{i}J)\acute{\gamma}^{\underline{12}} \end{bmatrix} = L \begin{bmatrix} (I + \mathrm{i}J)\gamma^{\underline{23}} \\ (I + \mathrm{i}J)\gamma^{\underline{31}} \\ I + \mathrm{i}J)\gamma^{\underline{12}} \end{bmatrix}
$$

where

$$
L = \begin{bmatrix}
1 + \dfrac{\phi^2}{2}(z_{\underline{1}})^2 & -\phi z_{\underline{3}} + \dfrac{\phi^2}{2} z_{\underline{1}} z_{\underline{2}} & \phi z_{\underline{2}} + \dfrac{\phi^2}{2} z_{\underline{1}} z_{\underline{3}} \\
\phi z_{\underline{3}} + \dfrac{\phi^2}{2} z_{\underline{2}} z_{\underline{1}} & 1 + \dfrac{\phi^2}{2}(z_{\underline{2}})^2 & -\phi z_{\underline{1}} + \dfrac{\phi^2}{2} z_{\underline{2}} z_{\underline{3}} \\
-\phi z_{\underline{2}} + \dfrac{\phi^2}{2} z_{\underline{3}} z_{\underline{1}} & \phi z_{\underline{1}} + \dfrac{\phi^2}{2} z_{\underline{3}} z_{\underline{2}} & 1 + \dfrac{\phi^2}{2} z_{\underline{2}} z_{\underline{3}}
\end{bmatrix}.
$$

Repeating previous arguments, we have $P = L^{-1}P'L$ where P' is defined in Eq. (10.78). Completing the computation we have

$$
P = v \begin{bmatrix}
1 - 3(v_{\underline{1}})^2 & -3v_{\underline{1}}v_{\underline{2}} & -3v_{\underline{1}}v_{\underline{3}} \\
-3v_{\underline{2}}v_{\underline{1}} & 1 - 3(v_{\underline{2}})^2 & -3v_{\underline{2}}v_{\underline{3}} \\
-3v_{\underline{3}}v_{\underline{1}} & -3v_{\underline{3}}v_{\underline{2}} & 1 - 3(v_{\underline{3}})^2
\end{bmatrix} \tag{10.83}
$$

where

$$
v_{\underline{k}} = \lambda_{\underline{k}} + \mathrm{i}\phi z_{\underline{k}} = \lambda_{\underline{k}} - \phi p_{\underline{k}} + \mathrm{i}\phi q_{\underline{k}}. \tag{10.84}
$$

From P we could determine the real and imaginary parts and then use those components to write out the curvature 2-forms. However, we can retain a greater simplicity if we apply the operator $I + \mathrm{i}J$ to the curvature 2-forms.

For example

$$\begin{aligned}
\mathscr{R}_{\underline{10}} &= \tfrac{1}{2}R_{\underline{10}ij}\hat{\gamma}^{\underline{ij}} \\
&= -(R^{\underline{10}}{}_{\underline{10}}\hat{\gamma}^{\underline{10}} + R^{\underline{10}}{}_{\underline{23}}\hat{\gamma}^{\underline{23}}) \\
&\quad - (R^{\underline{10}}{}_{\underline{20}}\hat{\gamma}^{\underline{20}} + R^{\underline{10}}{}_{\underline{31}}\hat{\gamma}^{\underline{31}}) - (R^{\underline{10}}{}_{\underline{30}}\hat{\gamma}^{\underline{30}} + R^{\underline{10}}{}_{\underline{12}}\hat{\gamma}^{\underline{12}}) \\
&= -(R^{\underline{10}}{}_{\underline{10}} + R^{\underline{10}}{}_{\underline{23}}J)\hat{\gamma}^{\underline{10}} - (R^{\underline{10}}{}_{\underline{20}} + R^{\underline{10}}{}_{\underline{31}}J)\hat{\gamma}^{\underline{20}} \\
&\quad - (R^{\underline{10}}{}_{\underline{30}} + R^{\underline{10}}{}_{\underline{12}}J)\hat{\gamma}^{\underline{30}}.
\end{aligned}$$

Since $(I + \mathrm{i}J)J = (I + \mathrm{i}J)(-\mathrm{i})$, we can write

$$\begin{aligned}
(I + \mathrm{i}J)\mathscr{R}_{\underline{10}} = {} & -(I + \mathrm{i}J)[(R^{\underline{10}}{}_{\underline{10}} - \mathrm{i}R^{\underline{10}}{}_{\underline{23}})\hat{\gamma}^{\underline{10}} \\
& + (R^{\underline{10}}{}_{\underline{20}} - \mathrm{i}R^{\underline{10}}{}_{\underline{31}})\hat{\gamma}^{\underline{20}} + (R^{\underline{10}}{}_{\underline{30}} - \mathrm{i}R^{\underline{10}}{}_{\underline{12}})\hat{\gamma}^{\underline{30}}].
\end{aligned}$$

A review of Section 10.1 shows that the coefficients of the $\hat{\gamma}^{\underline{k0}}$'s in this last equation are the components of the top row of the Petrov matrix. Thus

$$\begin{aligned}
(I + \mathrm{i}J)\mathscr{R}_{\underline{10}} &= -\nu(I + \mathrm{i}J)[\hat{\gamma}^{\underline{10}}(1 - 3(v_{\underline{1}})^2) + \hat{\gamma}^{\underline{20}}(-3v_{\underline{1}}v_{\underline{2}}) + \hat{\gamma}^{\underline{30}}(-3v_{\underline{1}}v_{\underline{3}})] \\
&= \nu(I + \mathrm{i}J)\{\hat{\gamma}^{\underline{k0}}(n_{\underline{1k}} + 3v_{\underline{1}}v_{\underline{k}})].
\end{aligned}$$

Generalizing

$$(I + \mathrm{i}J)\mathscr{R}_{\underline{j0}} = \nu(I + \mathrm{i}J)[\hat{\gamma}^{\underline{k0}}(n_{\underline{jk}} + 3v_{\underline{j}}v_{\underline{k}})]. \tag{10.85}$$

A similar calculation shows that

$$(I + \mathrm{i}J)\mathscr{R}_{\underline{23}} = \mathrm{i}\nu(I + \mathrm{i}J)[\hat{\gamma}^{\underline{k0}}(n_{\underline{1k}} + 3v_{\underline{1}}v_{\underline{k}})]. \tag{10.86}$$

At the beginning of this section, it was remarked that $\mathscr{R}_{\underline{jk}}$ is a polynomial in the constant m. Presumably this means that ν is a polynomial in m, that is

$$\nu = \nu_1 m + \nu_2 m^2 + \nu_3 m^3 + \nu_4 m^4.$$

Designating the lowest order term of $\mathscr{R}_{\underline{ij}}$ by $\overset{(1)}{\mathscr{R}}_{\underline{ij}}$ and replacing the function ν_1 by μ^3, we have

$$(I + \mathrm{i}J)\overset{(1)}{\mathscr{R}}_{\underline{j0}} = m\mu^3(I + \mathrm{i}J)\hat{\gamma}^{\underline{k0}}(n_{\underline{jk}} + 3v_{\underline{j}}v_{\underline{k}}), \tag{10.87}$$

$$(I + \mathrm{i}J)\overset{(1)}{\mathscr{R}}_{\underline{23}} = -\mathrm{i}m\mu^3(I + \mathrm{i}J)\hat{\gamma}^{\underline{k0}}(\delta^{\underline{1}}_{\underline{k}} + 3v^{\underline{1}}v_{\underline{k}}), \tag{10.88}$$

$$(I + \mathrm{i}J)\overset{(1)}{\mathscr{R}}_{\underline{31}} = -\mathrm{i}m\mu^3(I + \mathrm{i}J)\hat{\gamma}^{\underline{k0}}(\delta^{\underline{2}}_{\underline{k}} + 3v^{\underline{2}}v_{\underline{k}}), \tag{10.89}$$

$$(I + \mathrm{i}J)\overset{(1)}{\mathscr{R}}_{\underline{12}} = -\mathrm{i}m\mu^3(I + \mathrm{i}J)\hat{\gamma}^{\underline{k0}}(\delta^{\underline{3}}_{\underline{k}} + 3v^{\underline{3}}v_{\underline{k}}), \tag{10.90}$$

Except for a factor of 2, the operator $(I + \mathrm{i}J)$ is a projection operator. Since the image of a projection operator does not uniquely determine its preimage there are many alternative representations of $\overset{(1)}{\mathscr{R}}_{jk}$ to those that appear in Eqs. (10.87)–(10.90). However, for 2-vectors, the preimage of $(I + \mathrm{i}J)$ is unique if we require that the preimage be in the 3-dimensional space spanned by $\hat{\gamma}^{\underline{10}}$, $\hat{\gamma}^{\underline{20}}$, and $\hat{\gamma}^{\underline{30}}$.

The forms of the curvature tensor imposed by that of the Petrov matrices are not necessarily curvature tensors. A curvature 2-form must also satisfy the Bianchi identities. (See Appendix A.3.) In particular

$$\nabla_{[i}\mathscr{R}_{jk]} = 0. \tag{10.91}$$

Using these equations it is possible to determine the formula for the complex vector $\boldsymbol{\nu} = \hat{\gamma}^{\underline{k}} v_{\underline{k}}$ and then the formula for the vector $\boldsymbol{\lambda} = \hat{\gamma}^{\underline{k}} \lambda_{\underline{k}}$ and then the formula for the vector $\boldsymbol{\lambda} = \hat{\gamma}^{\underline{k}} \lambda_{\underline{k}}$.

To pursue this line of attack, we first note that

$$\nabla_i \mathscr{R}_{jk} = \frac{\partial}{\partial x^i}\mathscr{R}_{jk} - \boldsymbol{\Gamma}_i \mathscr{R}_{jk} + \mathscr{R}_{jk}\boldsymbol{\Gamma}_i$$

and

$$\boldsymbol{\Gamma}_i = -\frac{m}{2}\gamma^{pq}\frac{\partial}{\partial x^p}(w_i w_q).$$

Thus if we project out the lowest power of m, Eq. (10.91) becomes

$$\frac{\partial}{\partial x}{}_{[\underline{i}}\overset{(1)}{\mathscr{R}}_{\underline{jk}]} = 0. \tag{10.92}$$

For the Kerr metric all time derivatives are zero. Thus Eq. (10.92) implies that

$$\frac{\partial}{\partial x^{\underline{j}}}\overset{(1)}{\mathscr{R}}_{\underline{k0}} = \frac{\partial}{\partial x^{\underline{k}}}\overset{(1)}{\mathscr{R}}_{\underline{j0}}, \tag{10.93}$$

and

$$\frac{\partial}{\partial x^{\underline{1}}}\overset{(1)}{\mathscr{R}}_{\underline{23}} + \frac{\partial}{\partial x^{\underline{2}}}\overset{(1)}{\mathscr{R}}_{\underline{31}} + \frac{\partial}{\partial x^{\underline{3}}}\overset{(1)}{\mathscr{R}}_{\underline{12}} = 0. \tag{10.94}$$

From Eqs. (10.87) and (10.93), we have

$$\frac{\partial}{\partial x^{\underline{j}}}[\mu^3 n_{\underline{kp}} + 3\mu^3 v_{\underline{k}} v_{\underline{p}}] = \frac{\partial}{\partial x^{\underline{k}}}[\mu^3 n_{\underline{jp}} + 3\mu^3 v_{\underline{j}} v_{\underline{p}}].$$

Computing the derivatives, we have

$$3n_{\underline{kp}}\mu^2 \frac{\partial \mu}{\partial x^{\underline{j}}} + 9\mu^2 \frac{\partial \mu}{\partial x^{\underline{j}}} v_{\underline{k}} v_{\underline{p}} + 3\mu^3 \frac{\partial v_{\underline{k}}}{\partial x^{\underline{j}}} v_{\underline{p}} + 3\mu^3 v_{\underline{k}} \frac{\partial v_{\underline{p}}}{\partial x^{\underline{j}}}$$

$$= 3n_{\underline{jp}}\mu^2 \frac{\partial \mu}{\partial x^{\underline{k}}} + 9\mu^2 \frac{\partial \mu}{\partial x^{\underline{k}}} v_{\underline{j}} v_{\underline{p}} + 3\mu^3 \frac{\partial v_{\underline{j}}}{\partial x^{\underline{k}}} v_{\underline{p}} + 3\mu^3 v_{\underline{j}} \frac{\partial v_{\underline{p}}}{\partial x^{\underline{k}}}. \qquad (10.95)$$

From Eq. (10.84), it is clear that $v^{\underline{p}} v_{\underline{p}} = -1$ and

$$v^{\underline{p}} \frac{\partial}{\partial x^{\underline{n}}} v_{\underline{p}} = v^{\underline{p}} n_{\underline{pq}} \frac{\partial}{\partial x^{\underline{n}}} v^{\underline{q}} = \frac{1}{2} \frac{\partial}{\partial x^{\underline{n}}} (v^{\underline{p}} n_{\underline{pq}} v^{\underline{q}})$$

$$= \frac{1}{2} \frac{\partial}{\partial x^{\underline{n}}} (-1) = 0.$$

Therefore if we multiply both sides of Eq. (10.95) by $v^{\underline{p}}$ and carry out the indicated sums, we have

$$3v_{\underline{k}}\mu^2 \frac{\partial \mu}{\partial x_{\underline{j}}} - 9\mu^2 \frac{\partial \mu}{\partial x^{\underline{j}}} v_{\underline{k}} - 3\mu^3 \frac{\partial v_{\underline{k}}}{\partial x^{\underline{j}}} = 3v_{\underline{j}}\mu^2 \frac{\partial \mu}{\partial x^{\underline{k}}} - 9\mu^2 \frac{\partial \mu}{\partial x^{\underline{k}}} v_{\underline{j}} - 3\mu^3 \frac{\partial v_{\underline{j}}}{\partial x^{\underline{k}}}.$$

Dividing by -3μ and collecting like terms, we have

$$2\mu \frac{\partial \mu}{\partial x^{\underline{j}}} v_{\underline{k}} + \mu^2 \frac{\partial v_{\underline{k}}}{\partial x^{\underline{j}}} = 2\mu \frac{\partial \mu}{\partial x^{\underline{k}}} v_{\underline{j}} + \mu^2 \frac{\partial v_{\underline{j}}}{\partial x^{\underline{k}}}$$

or

$$\frac{\partial}{\partial x^{\underline{j}}} (\mu^2 v_{\underline{k}}) = \frac{\partial}{\partial x^{\underline{k}}} (\mu^2 v_{\underline{j}}).$$

This means that

$$\gamma^{\underline{jk}} \frac{\partial}{\partial x^{\underline{j}}} (\mu^2 v_{\underline{k}}) = \mathbf{d}(\mu^2 \mathbf{v}) = 0.$$

In turn this implies that at least locally there exists some scalar function η such that

$$\mu^2 \mathbf{v} = \mathbf{d}\eta. \qquad (10.96)$$

The remaining Bianchi identity comes from Eqs. (10.94), (10.88), (10.89), and (10.90). From these equations, we have

$$\frac{\partial}{\partial x^{\underline{j}}} [\mu^3 \delta^{\underline{j}}_{\underline{k}} + 3\mu^3 v^{\underline{j}} v_{\underline{k}}] = 0$$

or

$$3\mu^2 \frac{\partial \mu}{\partial x^{\underline{k}}} + 9\mu^2 \frac{\partial u}{\partial x^{\underline{j}}} v^{\underline{j}} v_{\underline{k}} + 3\mu^3 \frac{\partial v^{\underline{j}}}{\partial x^{\underline{j}}} v_{\underline{k}} + 3\mu^3 v^{\underline{j}} \frac{\partial v_{\underline{k}}}{\partial x^{\underline{j}}} = 0.$$

Multiplying this last equation by $-(1/3\mu)v^{\underline{k}}$, and then collecting like terms gives us

$$2\mu v^{\underline{j}} \frac{\partial \mu}{\partial x^{\underline{j}}} + \mu^2 \frac{\partial v^{\underline{j}}}{\partial x^{\underline{j}}} = 0$$

or

$$\frac{\partial}{\partial x^{\underline{j}}}(\mu^2 v^{\underline{j}}) = 0. \tag{10.97}$$

This means that

$$\boldsymbol{\delta}(\mu^2 \mathbf{v}) = \boldsymbol{\delta}\mathbf{d}(\eta) = \boldsymbol{\nabla}^2 \eta = 0. \tag{10.98}$$

As in Chapter 9,

$$\boldsymbol{\nabla}^2 = \left(\gamma^{\underline{k}} \frac{\partial}{\partial x^{\underline{k}}}\right)^2 = -\left(\frac{\partial^2}{(\partial x^{\underline{1}})^2}, \frac{\partial^2}{(\partial x^{\underline{2}})^2}, \frac{\partial^2}{(\partial x^{\underline{3}})^2}\right).$$

For a radially symmetric solution,

$$\eta = -\frac{A^2}{r} + B, \tag{10.99}$$

$$\mu^2 v_{\underline{k}} = \frac{\partial \eta}{\partial x^{\underline{k}}} = \frac{A^2}{r^3} x^{\underline{k}}$$

or

$$v_{\underline{k}} = \frac{\mu^{-2} A^2 x^{\underline{k}}}{r^3}. \tag{10.100}$$

Since

$$1 = \sum_{\underline{k}=\underline{1}}^{\underline{3}} (v_{\underline{k}})^2 = \frac{\mu^{-4} A^4}{r^6} \sum_{\underline{k}=\underline{1}}^{\underline{3}} (x^{\underline{k}})^2 = \frac{\mu^{-4} A^4}{r^4},$$

we see that

$$\mu^2 = \frac{A^2}{r^2}. \tag{10.101}$$

Combining Eqs. (10.100) and (10.101), we have

$$v_{\underline{k}} = \lambda_{\underline{k}} = \frac{x^{\underline{k}}}{r}. \tag{10.102}$$

Clearly the constant B in Eq. (10.99) has no impact on the metric. The solution of Eq. (10.102) corresponds to a version of the Schwarzschild metric known as the Eddington form of the Schwarzschild solution (A. S. Eddington 1924). (See Problem 10.7.)

For the Kerr metric, we clearly need a complex solution for $v_{\underline{k}}$. If

$$\eta = -A^2[x^2 + y^2 + (z + ia)^2]^{-\frac{1}{2}} = -A^2(\rho - i\sigma)^{-1}, \tag{10.103}$$

then from Eq. (10.96)

$$\mu^2(v_1, v_2, v_3) = A^2(\rho - i\sigma)^{-3}(x, y, z + ia). \tag{10.104}$$

At this point you might infer that the sign conventions used for σ and a in Eqs. (10.103) and (10.104) represent a deviation from the sign conventions used in Chapter 9. Actually that is not the case. The sign conventions used above are consistent with those used in Chapter 9. Furthermore they are necessary to obtain the same answer for λ that was obtained in Chapter 9.

Returning to our computation, we note that

$$\begin{aligned}\sum_{\underline{k}=\underline{1}}^{3} (v_{\underline{k}})^2 = 1 &= \mu^{-4}A^4(\rho - i\sigma)^{-6}[x^2 + y^2 + (z + ia)^2] \\ &= \mu^{-4}A^4(\rho - i\sigma)^{-4}.\end{aligned}$$

Thus

$$A = \mu(\rho - i\sigma). \tag{10.105}$$

Using this last result, it is possible to eliminate the constant A from Eq. (10.104). We then have

$$\begin{aligned}\vec{v} = (v_{\underline{1}}, v_{\underline{2}}, v_{\underline{3}}) &= \left(\frac{x}{\rho - i\sigma}, \frac{y}{\rho - i\sigma}, \frac{z + ia}{\rho - i\sigma}\right) \\ &= \left(\frac{\rho x}{\rho^2 + \sigma^2}, \frac{\rho y}{\rho^2 + \sigma^2}, \frac{\rho z - \sigma a}{\rho^2 + \sigma^2}\right) \\ &\quad + i\left(\frac{\sigma x}{\rho^2 + \sigma^2}, \frac{\sigma y}{\rho^2 + \sigma^2}, \frac{\sigma z + \rho a}{\rho^2 + \sigma^2}\right).\end{aligned} \tag{10.106}$$

It is not difficult to extract $\vec{\lambda}$ from $\vec{v}$. Since $\vec{v} = \vec{\lambda} - \phi\vec{p} + i\phi\vec{q}$, we can take the

imaginary part of Eq. (10.106) to obtain

$$\phi\vec{q} = \left(\frac{\sigma x}{\rho^2 + \sigma^2}, \frac{\sigma y}{\rho^2 + \sigma^2}, \frac{\sigma z + \rho a}{\rho^2 + \sigma^2}\right). \tag{10.107}$$

Referring to Fig. 10.1 and using ordinary vector notation, we get

$$\vec{v} \times \phi\vec{q} = \phi\vec{\lambda} \times \vec{q} - \phi^2\vec{p} \times \vec{q} = \phi\vec{p} + \phi^2\vec{\lambda}.$$

Combining this equation with Eqs. (10.106) and (10.107) gives us

$$\phi\vec{p} + \phi^2\vec{\lambda} = \left(\frac{ay}{\rho^2 + \sigma^2}, \frac{-ax}{\rho^2 + \sigma^2}, 0\right). \tag{10.108}$$

Taking the real part of Eq. (10.106), we have

$$\vec{\lambda} - \phi\vec{p} = \left(\frac{\rho x}{\rho^2 + \sigma^2}, \frac{\rho y}{\rho^2 + \sigma^2}, \frac{\rho z - \sigma a}{\rho^2 + \sigma^2}\right). \tag{10.109}$$

Adding Eq. (10.108) and (10.109) gives us

$$(1 + \phi^2)\vec{\lambda} = \left(\frac{\rho x + ay}{\rho^2 + \sigma^2}, \frac{\rho y - ax}{\rho^2 + \sigma^2}, \frac{\rho z - \sigma a}{\rho^2 + \sigma^2}\right). \tag{10.110}$$

Using the fact that the length of $\vec{\lambda}$ is 1, it is now possible to obtain $\vec{\lambda}$ from Eq. (10.110). Carrying out the computation, one gets

$$\vec{\lambda} = (\lambda_{\underline{1}}, \lambda_{\underline{2}}, \lambda_{\underline{3}}) = \left(\frac{\rho x + ay}{\rho^2 + a^2}, \frac{\rho y - ax}{\rho^2 + a^2}, \frac{\rho z - \sigma a}{\rho^2 + a^2}\right). \tag{10.111}$$

Having obtained a formula for $\vec{\lambda}$, what remains is the task of determining w_0. Now that we have obtained an explicit form for $v_{\underline{k}}$, it is possible to simplify Eq. (10.87). To do this, we first note that

$$\begin{aligned}\frac{\partial}{\partial x^{\underline{k}}}\frac{1}{\rho - \mathrm{i}\sigma} &= \frac{\partial}{\partial x^{\underline{k}}}[x^2 + y^2 + (z + \mathrm{i}a)^2]^{-\frac{1}{2}}\\ &= -[x^2 + y^2 + (z + \mathrm{i}a)^2]^{-\frac{3}{2}}u_k,\end{aligned}$$

where $u_1 = x$, $u_2 = y$, and $u_3 = z + \mathrm{i}a$. Furthermore,

$$\begin{aligned}\frac{\partial^2}{\partial x^{\underline{j}}\,\partial x^{\underline{k}}}\frac{1}{\rho - \mathrm{i}\sigma} &= [x^2 + y^2 + (z + \mathrm{i}a)^2]^{-\frac{3}{2}}n_{jk}\\ &\quad + 3[x^2 + y^2 + (z + \mathrm{i}a)^2]^{-\frac{5}{2}}u_j u_k\\ &= (\rho - \mathrm{i}\sigma)^{-3}(n_{\underline{jk}} + 3v_{\underline{j}}v_{\underline{k}}).\end{aligned} \tag{10.112}$$

Using this result, along with Eq. (10.105), Eq. (10.87) becomes

$$(I + iJ)\overset{(1)}{\mathscr{R}}_{\underline{j0}} = mA^3(I + iJ)\hat{\gamma}^{\underline{k0}} \frac{\partial^2}{\partial x^{\underline{j}}\,\partial x^{\underline{k}}} \frac{1}{\rho - i\sigma}.$$

One can extract $\overset{(1)}{\mathscr{R}}_{\underline{j0}}$ from this last equation by taking the real part of both sides. In this fashion, we get

$$\begin{aligned}\overset{(1)}{\mathscr{R}}_{\underline{j0}} &= mA^3\hat{\gamma}^{\underline{k0}} \frac{\partial^2}{\partial x^{\underline{j}}\,\partial x^{\underline{k}}} \frac{\rho}{\rho^2 + \sigma^2} \\ &\quad - mA^3J\hat{\gamma}^{\underline{k0}} \frac{\partial^2}{\partial x^{\underline{j}}\,\partial x^{\underline{k}}} \frac{\sigma}{\rho^2 + \sigma^2}.\end{aligned} \tag{10.113}$$

On the other hand

$$\tfrac{1}{2}\mathscr{R}_{j0} = \frac{\partial}{\partial x^j}\mathbf{\Gamma}_0 + \mathbf{\Gamma}_0\mathbf{\Gamma}_j - \mathbf{\Gamma}_j\mathbf{\Gamma}_0,$$

where

$$\mathbf{\Gamma}_k = -\frac{m}{2}\gamma^{pq}\frac{\partial}{\partial x^p}(w_k w_q).$$

If we use the fact that

$$\gamma^r = \hat{\gamma}^{\underline{r}} + mw^r w_s \hat{\gamma}^{\underline{s}}$$

and then retain the lowest power of m, we get

$$\begin{aligned}\overset{(1)}{\mathscr{R}}_{\underline{j0}} &= -m\hat{\gamma}^{\underline{kq}} \frac{\partial^2}{\partial x^{\underline{j}}\,\partial x^{\underline{k}}}(w_0 w_q) \\ &= -m\hat{\gamma}^{\underline{k0}} \frac{\partial^2}{\partial x^{\underline{j}}\,\partial x^{\underline{k}}}(w_0)^2 - m\sum_{\underline{q}\neq\underline{0}} \hat{\gamma}^{\underline{kq}} \frac{\partial^2}{\partial x^{\underline{j}}\,\partial x^{\underline{k}}}(w_0 w_{\underline{q}})\end{aligned}$$

Matching this with Eq. (10.113) gives us

$$\frac{\partial^2}{\partial x^{\underline{j}}\,\partial x^{\underline{k}}}(w_0)^2 = -A^3 \frac{\partial^2}{\partial x^{\underline{j}}\,\partial x^{\underline{k}}} \frac{\rho}{\rho^2 + \sigma^2}.$$

This implies that

$$\frac{\partial}{\partial x^{\underline{k}}}(w_0)^2 = -A^3 \frac{\partial}{\partial x^{\underline{k}}} \frac{\rho}{\rho^2 + \sigma^2} + \text{a constant}.$$

However, if $(w_0)^2 = 0$ at ∞ then the constant must be zero. Repeating the

same argument, we have

$$(w_0)^2 = -A^3 \frac{\rho}{\rho^2 + \sigma^2}.$$

The choice of a numerical value of A determines the physical interpretation of m. Following the standard convention used in Chapter 9, $A = -1$. Therefore we have

$$(w_0)^2 = \frac{\rho}{\rho^2 + \sigma^2}.$$

Thus we see how the Petrov classification scheme and the symmetry of the principal null vectors can be exploited to obtain a metric.

Problem 10.5. Show that the null rotation of Eq. (10.80) preserves the null vector $\hat{\gamma}^{\underline{0}} + \hat{\gamma}^{\underline{k}}\lambda_{\underline{k}}$ but not the null vector $\hat{\gamma}^{\underline{0}} - \hat{\gamma}^{\underline{k}}\lambda_{\underline{k}}$.

Problem 10.6. Show that the null rotation of Eq. (10.80) can be factored into a product of a boost and a spatial rotation. Hint: $\mathscr{L}$ may be factored into the product $\mathscr{B}\mathscr{R}$ or $\mathscr{R}\mathscr{B}'$. The boost operator is different in the two cases but in either case the axis of rotation is (q_1, q_2, q_3).

Problem 10.7. The line element for the usual Schwarzschild metric is

$$(\mathrm{d}s)^2 = \left(1 - \frac{2m}{r}\right) c^2(\mathrm{d}t)^2 - \left(1 - \frac{2m}{r}\right)^{-1} (\mathrm{d}r)^2 - r^2[(\mathrm{d}\theta)^2 + r^2 \sin\theta\,(\mathrm{d}\phi)^2].$$

Suppose one substitutes

$$ct = x^0 - 2m \ln \left|\frac{r}{2m} - 1\right|, \qquad r = \bar{r}, \qquad \theta = \bar{\theta}, \qquad \text{and} \qquad \phi = \bar{\phi}.$$

Show that the resulting line element is

$$\begin{aligned}(\mathrm{d}s)^2 &= (\mathrm{d}x^0)^2 - (\mathrm{d}\bar{r})^2 - \bar{r}^2[(\mathrm{d}\bar{\theta})^2 + \bar{r}^2 \sin^2\bar{\theta}(\mathrm{d}\bar{\phi})^2] - \frac{2m}{\bar{r}}[\mathrm{d}x^0 + \mathrm{d}\bar{r}]^2 \\ &= (\mathrm{d}x^0)^2 - (\mathrm{d}x)^2 - (\mathrm{d}y)^2 - (\mathrm{d}z)^2 - \frac{2m}{r}\left[\mathrm{d}x^0 + \frac{\mathrm{d}x}{r} + \frac{\mathrm{d}y}{r} + \frac{\mathrm{d}z}{r}\right]^2.\end{aligned}$$

This is Eddington's form of the Schwarzshild metric (Eddington 1924).

Problem 10.8. From Eq. (10.111), we know the forula for $\boldsymbol{\lambda} = \hat{\gamma}^{\underline{k}}\lambda_{\underline{k}}$ in Cartesian coordinates. Convert this to oblate spheroidal coordinates to show that this is the same result obtained in Chapter 9.

11

MATRIX REPRESENTATIONS AND CLASSIFICATIONS OF CLIFFORD ALGEBRAS

11.1 Matrix Representations of Clifford Algebras

In the first nine chapters, the intimate connection between differential geometry and Clifford algebra has been emphasized. However, some physicists have found applications for Clifford algebras which are virtually divorced from any underlying geometry. For example, in his book *Lie Algebras in Particle Physics*, Howard Georgi uses elements of a Clifford algebra to construct creation and annihilation operators (1982, pp. 209–219).

In any unified field theory, the required symmetries are more apparent than any geometric manifold that some theoretical physicist may introduce to "explain" the symmetries. If Clifford algebras are found to be useful in such applications, it is important for the theoretician to be sensitive to the fact that different metric spaces may correspond to isomorphic Clifford algebras.

For example, suppose we designate the signature of a pseudo-Euclidean space by (p, q). That is $(\hat{\gamma}_1)^2 = (\hat{\gamma}_2)^2 = \cdots = (\hat{\gamma}_p)^2 = I$ and

$$(\hat{\gamma}_{p+1})^2 = (\hat{\gamma}_{p+2})^2 = \cdots = (\hat{\gamma}_{p+q})^2 = -I.$$

Suppose, in addition, that we designate the corresponding universal Clifford algebra by $\boldsymbol{R}_{p,q}$. Then $\boldsymbol{R}_{2,0}$ is isomorphic to $\boldsymbol{R}_{1,1}$. To see this, we note that for $\boldsymbol{R}_{2,0}$ we can represent $\hat{\gamma}_1$ by σ_3, and $\hat{\gamma}_2$ by σ_1. In this representation $\boldsymbol{R}_{2,0}$, treated as a vector space, is spanned by $\{I, \hat{\gamma}_1, \hat{\gamma}_2, \hat{\gamma}_{12}\} = \{I, \sigma_3, \sigma_1, \mathrm{i}\sigma_2\}$. On the other hand, for $\boldsymbol{R}_{1,1}$ we can represent $\hat{\gamma}_1$ by σ_3 and $\hat{\gamma}_2$ by $\mathrm{i}\sigma_2$. In this representation, $\boldsymbol{R}_{1,1}$ is spanned by $\{I, \hat{\gamma}_1, \hat{\gamma}_2, \hat{\gamma}_{12}\} = \{I, \sigma_3, \mathrm{i}\sigma_2, \sigma_1\}$. In these representations, $\boldsymbol{R}_{2,0}$ and $\boldsymbol{R}_{1,1}$ are both spanned by the same matrices and are thus isomorphic to one another.

It is not too difficult to convince oneself that an arbitrary real 2×2 matrix can be written as a linear combination of I, σ_3, $\mathrm{i}\sigma_2$, and σ_1. Thus both $\boldsymbol{R}_{2,0}$ and $\boldsymbol{R}_{1,1}$ are isomorphic to the algebra of real 2×2 matrices, which is designated by $\boldsymbol{R}(2)$.

This does not mean that all Clifford algebras with the same dimension are isomorphic to one another. For example. $\boldsymbol{R}_{0,2}$ is not isomorphic to $\boldsymbol{R}(2)$. One can represent $\boldsymbol{R}_{0,2}$ by a subalgebra of the complex 2×2 matrices or by a subalgebra of the real 4×4 matrices. However, it is simpler to identify $\boldsymbol{R}_{0,2}$ as the algebra of quaternions, that is $\{I, \hat{\gamma}_1, \hat{\gamma}_2, \hat{\gamma}_{12}\} = \{I, \boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$. The algebra of quaternions is usually designated by $\boldsymbol{H}$. To summarize the last few paragraphs, we have

$$\boldsymbol{R}_{2,0} = \boldsymbol{R}_{1,1} = \boldsymbol{R}(2) \qquad \text{(the real } 2 \times 2 \text{ matrices)} \tag{11.1}$$

and

$$\boldsymbol{R}_{0,2} = \boldsymbol{H} \qquad \text{(the quaternions).} \tag{11.2}$$

It is not too difficult to construct similar formulas for all Clifford algebras. This will be done for complex Clifford algebras in this section and real Clifford algebras in the next section. Meanwhile, the construction of representations of universal Clifford algebras is greatly facilitated by a theorem introduced by Marcel Riesz (1993, pp. 10–12). His original theorem rephrased in current terminology states that all Clifford algebras are universal Clifford algebras. It is now recognized that his theorem is false. Correctly stated, the theorem should state that most Clifford algebras are universal. In the proof of his theorem, Marcel Riesz overlooked an exceptional case. Nonetheless the method used by Marcel Riesz is useful to prove the correct version.

Before proving the Riesz theorem, let us review the pertinent definitions. An n-dimensional orthonormal system of Dirac matrices $\hat{\gamma}_1, \hat{\gamma}_2, \ldots, \hat{\gamma}_n$ has the following properties:

(1) $(\hat{\gamma}_k)^2 = I$ for $k = 1, 2, \ldots, p$ and $(\hat{\gamma}_k)^2 = -I$ for

$$k = p + 1, p + 2, \ldots, p + q = n.$$

(2) $\hat{\gamma}_j\hat{\gamma}_k + \hat{\gamma}_k\hat{\gamma}_j = 0$ for $k \neq j$.

The algebra generated by such a system of Dirac matrices is called a *Clifford algebra*. If the field of scalars is the field of real numbers, the algebra is said to be a *real Clifford algebra*. If the field of scalars is the field of complex numbers, the algebra is said to be a *complex Clifford algebra*.

Besides properties (1) and (2), there is an additional property which occurs for most systems of Dirac matrices. This is:

(3) by taking all possible products of the n Dirac matrices, one can form a set of 2^n linearly independent matrices. (These products may be written in the form $M_1M_2 \ldots M_n$ where $M_k = \hat{\gamma}_k$ or I.)

If property (3) holds, the resulting Clifford algebra is said to be *universal*. The universal Clifford algebra corresponding to the signature (p, q) will be designated by $\boldsymbol{R}_{p,q}(\boldsymbol{C}_{p,q})$ if the field of scalars is real (complex).

To determine whether or not a system of Dirac matrices generates a universal Clifford algebra, the corrected version of Marcel Riesz's theorem is extremely useful.

Theorem 11.1. An n-dimensional orthonormal system of Dirac matrices generates a vector space (and therefore an algebra) of dimension 2^n unless the product $\hat{\gamma}_1\hat{\gamma}_2 \ldots \hat{\gamma}_n$ is a scalar multiple of I. (For real Clifford algebras, the exceptional case occurs when $J = \hat{\gamma}_{12\ldots n} = \pm I$. For complex Clifford algebras, the exceptional case occurs when $J = \pm I$ or $\pm iI$.)

Proof. With one exception, a product of one or more distinct Dirac matrices will anticommute with at least one Dirac matrix. To see this, we note that a product of an even number of Dirac matrices will anticommute with any Dirac matrix appearing in the product. We also see that the product of an odd number of Dirac matrices will anticommute with any Dirac matrix which does not appear in the product. The one product which commutes with all Dirac matrices is the product $\hat{\gamma}_{12\ldots n}$ where n is odd. The proof now proceeds by self-contradiction. Suppose the products are not linearly independent. In that case there exists a set of coefficients $A^{j_1 j_2 \ldots j_k}$ (not all zero) such that

$$\sum_{k=0}^{n} \sum_{j_1 < j_2 \ldots < j_k} A^{j_1 j_2 \ldots j_n} \hat{\gamma}_{j_1 j_2 \ldots j_k} = 0. \tag{11.3}$$

If the coefficient of I in Eq. (11.3) is not zero then one can divide Eq. (11.3) by that coefficient and obtain the equation

$$I + \sum B^{j_1 j_2 \ldots j_k} \hat{\gamma}_{j_1 j_2 \ldots j_k} = 0 \tag{11.4}$$

where the sum does not include the identity matrix. If the coefficient of I in Eq. (11.3) is zero, one can pick out a term with a non-zero coefficient (say $\hat{\gamma}_{m_1 m_2 \ldots m_k}$) and multiply the equation by $(\hat{\gamma}_{m_1 m_2 \ldots m_k})^{-1}$ which equals $\pm\hat{\gamma}_{m_1 m_2 \ldots m_k}$. In this fashion one can always obtain an equation of the same form as Eq. (11.4) from Eq. (11.3).

If the $B^{j_1 j_2 \ldots j_k}$'s are all zero, we already have the desired contradiction. If the sum in Eq. (11.4) contains a product (say $\hat{\gamma}_{m_1 m_2 \ldots m_k}$) which anticommutes with $\hat{\gamma}_m$ then one can multiply Eq. (11.4) on the left by $\hat{\gamma}_m$ and on the right by $(\hat{\gamma}_m)^{-1}$ and obtain

$$I + \sum B^{j_1 j_2 \ldots j_k} \hat{\gamma}_m \hat{\gamma}_{j_1 j_2 \ldots j_k} (\hat{\gamma}_m)^{-1} = 0. \tag{11.5}$$

We note that

$$\hat{\gamma}_m \hat{\gamma}_{m_1 m_2 \ldots m_k} (\hat{\gamma}_m)^{-1} = -\hat{\gamma}_{m_1 m_2 \ldots m_k} \hat{\gamma}_m (\hat{\gamma}_m)^{-1} = -\hat{\gamma}_{m_1 m_2 \ldots m_k}.$$

Therefore we can add Eq. (11.5) to Eq. (11.4) and thereby obtain a new

equation which, except for a factor of 2, is identical in form to Eq. (11.4) except that at least one less term ($\hat{\gamma}_{m_1 m_2 \ldots m_k}$) will now appear in the sum.

If n is even, this process can be continued until the sum reduces to zero and thus obtain the contradiction $I = 0$. If n is odd the process can be continued, until we have

$$I + \alpha J = 0.$$

Thus we see that our desired contradiction occurs unless J is a scalar multiple of I and so the theorem is proved. ∎

The class of possible nonuniversal Clifford algebras is narrowed by the following theorem.

Theorem 11.2. If the Clifford algebra generated by an n-dimensional system of orthonormal Dirac matrices is not universal then n is odd. Furthermore, if the Clifford algebra is real and not universal then $p - q - 1$ is an integral multiple of 4.

Proof. The first sentence follows from the proof of Theorem 11.1. The second sentence follows from the requirement that $J^2 = I$. To see this, we first note that if $J = \hat{\gamma}_1 \hat{\gamma}_2 \hat{\gamma}_3 \ldots \hat{\gamma}_n$, then

$$\begin{aligned} J &= (-1)^{n-1} \hat{\gamma}_n \hat{\gamma}_1 \hat{\gamma}_2 \hat{\gamma}_3 \cdots \hat{\gamma}_{n-1} \\ &= (-1)^{(n-1)+(n-2)} \hat{\gamma}_n \hat{\gamma}_{n-1} \hat{\gamma}_1 \hat{\gamma}_2 \hat{\gamma}_3 \cdots \hat{\gamma}_{n-3} \\ &= (-1)^{\frac{1}{2}n(n-1)} \hat{\gamma}_n \hat{\gamma}_{n-1} \cdots \hat{\gamma}_3 \hat{\gamma}_2 \hat{\gamma}_1 . \end{aligned}$$

Thus

$$J^2 = (-1)^{\frac{1}{2}n(n-1)} (\hat{\gamma}_1 \hat{\gamma}_2 \hat{\gamma}_3 \cdots \hat{\gamma}_n)(\hat{\gamma}_n \hat{\gamma}_{n-1} \cdots \hat{\gamma}_3 \hat{\gamma}_2 \hat{\gamma}_1)$$

or

$$J^2 = (-1)^{\frac{1}{2}n(n-1)}(-1)^q I = (-1)^{\frac{1}{2}n(n-1)}(-1)^{-q} I. \tag{11.6}$$

Since $J^2 = I$,

$$\tfrac{1}{2}n(n-1) - q = 2k \qquad \text{for some integer } k.$$

Since n is odd, $n = 2m + 1$ for some integer m, this last equation becomes

$$(2m+1)(2m) - 2q = 4k \qquad \text{or} \qquad 2m - 2q = 4(k - m^2).$$

Since $2m = n - 1 = p + q - 1$, this last equation becomes $p - q - 1 = 4(k - m^2)$. ∎

After a few more paragraphs, it will be shown that for pseudo-Euclidean spaces of any finite dimension with any signature, it is always possible to

construct a system of Dirac matrices such that $\hat{\gamma}_{123\ldots n}$ is not a scalar multiple of I. This does not mean that nonuniversal Clifford algebras do not exist.

To see this, suppose the system $\{\hat{\gamma}_1, \hat{\gamma}_2, \ldots, \hat{\gamma}_{2m}\}$ generates a universal Clifford algebra. Then if we define $\hat{\gamma}_{2m+1} = \hat{\gamma}_1\hat{\gamma}_2 \ldots \hat{\gamma}_{2m}$, it is not difficult to convince oneself that the system $\{\hat{\gamma}_1, \hat{\gamma}_2, \ldots, \hat{\gamma}_{2m}, \hat{\gamma}_{2m+1}\}$ generates a nonuniversal Clifford algebra. It can be shown that any nonuniversal Clifford algebra can be constructed in this manner. (See Problem 11.1.) This means that any nonuniversal Clifford algebra associated with a Euclidean or pseudo-Euclidean space of dimension n (or $2m + 1$) is isomorphic to a universal Clifford algebra associated with a Euclidean or pseudo-Euclidean space of dimension $n - 1$.

We are now in a good position to consider the problem of constructing matrix representations for universal Clifford algebras. Probably the easiest way to construct explicit representations is by using the *Kronecker product* of matrices. Suppose A is an $n \times n$ matrix and B is an $m \times m$ matrix. The Kronecker product $A \circ B$ is an $mn \times mn$ matrix. In particular if $A = [a_{ij}]$, then the Kronecker product is defined by the partitioned matrix:

$$A \circ B = \begin{bmatrix} a_{11}B & a_{12}B & \ldots & a_{1n}B \\ a_{21}B & a_{22}B & \ldots & a_{2n}B \\ \vdots & \vdots & & \vdots \\ a_{n1}B & a_{n2}B & \ldots & a_{nn}B \end{bmatrix}. \tag{11.7}$$

For example, if

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

then

$$A \circ B = \begin{bmatrix} a_{11}b_{11} & a_{11}b_{12} & a_{12}b_{11} & a_{12}b_{12} \\ a_{11}b_{21} & a_{11}b_{22} & a_{12}b_{21} & a_{12}b_{22} \\ a_{21}b_{11} & a_{21}b_{12} & a_{22}b_{11} & a_{22}b_{12} \\ a_{21}b_{21} & a_{21}b_{22} & a_{22}b_{21} & a_{22}b_{22} \end{bmatrix}.$$

The properties of the Kronecker product are thoroughly discussed in a book entitled *Kronecker Products and Matrix Calculus with Applications* by Alexander Graham (1981).

It is not too difficult to show that

$$(A \circ B)(C \circ D) = AC \circ BD \tag{11.8}$$

where $(A \circ B)(C \circ D)$ represents the ordinary matrix product of $(A \circ B)$ with

$(C \circ D)$ and in a similar fashion AC and BD are respectively the ordinary matrix products of A with C and B with D.

Furthermore, it can be shown that

$$A \circ (B \circ C) = (A \circ B) \circ C. \tag{11.9}$$

Now we have the machinery to construct explicit matrix representations of Dirac matrices for a nondegenerate vector space over either the field of real or complex scalars. Actually we will construct matrix representations of the Dirac matrices in $\boldsymbol{C}_{p,q}$ and then note that the same Dirac matrices can be used for $\boldsymbol{R}_{p,q}$. The type of Clifford algebras which will be treated first are of the type $\boldsymbol{C}_{p,0}$ where p is an even integer.

To carry out this construction, we will use the Pauli spin matrices:

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2 = \begin{bmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{bmatrix}, \quad \text{and} \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

For $\boldsymbol{C}_{2,0}$, we can use $\hat{\gamma}_1 = \sigma_1$ and $\hat{\gamma}_2 = \sigma_2$. For $\boldsymbol{C}_{4,0}$, we can let $\hat{\gamma}_1 = \sigma_1 \circ \sigma_1$, $\hat{\gamma}_2 = \sigma_1 \circ \sigma_2$, $\hat{\gamma}_3 = \sigma_1 \circ \sigma_3$, and $\hat{\gamma}_4 = \sigma_2 \circ I$.

This process can be continued by induction. Suppose it is possible to construct a matrix representation of size $2^m \times 2^m$ for $\boldsymbol{C}_{2m,0}$. Suppose also that we designate the matrix representation of the Dirac matrices for this space by

$$\hat{\gamma}_k(2m) \quad \text{for } k = 1, 2, \ldots, 2m.$$

We can then obtain a matrix representation for the Dirac matrices of $\boldsymbol{C}_{2m+2,0}$ as follows:

$$\hat{\gamma}_k(2m+2) = \sigma_1 \circ \hat{\gamma}_k(2m) \quad \text{for } k = 1, 2, \ldots, 2m, \tag{11.10}$$

$$\hat{\gamma}_{2m+1}(2m+2) = \mathrm{i}^m \sigma_1 \circ J(2m), \tag{11.11}$$

$$\hat{\gamma}_{2m+2}(2m+2) = \sigma_2 \circ I, \tag{11.12}$$

where

$$J(2m) = \hat{\gamma}_1(2m)\hat{\gamma}_2(2m) \ldots \hat{\gamma}_{2m}(2m).$$

(The factor i^m that appears in Eq. 11.11) has been chosen to guarantee that $(\hat{\gamma}_{2m+1}(2m+2))^2 = +I$.)

It is not difficult to show that the system of matrices just constructed is indeed an orthonormal system of Dirac matrices, where $(\hat{\gamma}_k)^2 = +I$ for $k = 1, 2, \ldots, 2m+2$. (See Prob. 11.2.) From Theorem (11.1), this is all that is required to generate the universal Clifford algebra $\boldsymbol{C}_{2m+2,0}$ since $2m+2$ is an even integer.

To construct the Dirac matrices for universal Clifford algebras of the

type $C_{p,q}$ where $p + q = 2m$ is now a simple matter. One can take the Dirac matrices constructed for $C_{2m,0}$ and simply leave the first p matrices unchanged and then multiply each of the remaining q Dirac matrices by i.

If $p + q = 2m$, the elements of $C_{p,q}$ form a vector space of dimension 2^{2m}. Over the field of complex numbers, the set of complex matrices of size $2^m \times 2^m$ also form a vector space of dimension 2^{2m}. Thus it is clear from the above construction that if $p + q = 2m$, the Clifford algebra $C_{p,q}$ is isomorphic to $C(2^m)$ where $C(2^m)$ designates the algebra of complex matrices of size $2^m \times 2^m$.

It turns out that all complex universal Clifford algebras for nondegenerate pseudo-Euclidean spaces of a given dimension are the same regardless of the signature of the metric. We have just shown that to be the case when the dimension of the pseudo-Euclidean space is even. It will be left to you to show that if $p + q = 2m + 1$ then $C_{p,q}$ is isomorphic to ${}^2C(2^m)$ which designates the direct sum $C(2^m) \oplus C(2^m)$. (Problem 11.3.)

For real Clifford algebras, the situation is considerably more complicated. The classification of these algebras will be discussed in detail in the next section.

Problem 11.1. Suppose the system $\{\hat{\gamma}_1, \hat{\gamma}_2, \ldots, \hat{\gamma}_{2m+1}\}$ generates a nonuniversal Clifford algebra. Show that if any single Dirac matrix is eliminated from this system then the resulting system of $2m$ Dirac matrices generates a universal Clifford algebra which is isomorphic to the given nonuniversal Clifford algebra. (This shows that if (p, q) is the signature associated with a real (complex) nonuniversal Clifford algebra then that nonuniversal Clifford algebra is isomorphic to the universal algebra $R_{p-1,q}$ $(C_{p-1,q})$ if $p \neq 0$. The same nonuniversal algebra is also isomorphic to $R_{p,q-1}$ $(C_{p,q-1})$ if $q \neq 0$.)

Problem 11.2. Show that

$$\hat{\gamma}_k(2m + 2)\hat{\gamma}_j(2m + 2) + \hat{\gamma}_j(2m + 2)\hat{\gamma}_k(2m + 2) = 2\delta_{jk}I,$$

where $\hat{\gamma}_k(2m + 2)$ is defined by Eqs. (11.10)–(11.12).

Problem 11.3. Suppose the Dirac matrices for $C_{2m,0}$ are designated by $\hat{\gamma}_k(2m)$ for $k = 1, 2, \ldots, 2m$. Then define

$$\hat{\gamma}_k(2m + 1) = \sigma_3 \circ \hat{\gamma}_k(2m) \quad \text{for } k = 1, 2, \ldots, 2m, \tag{11.13}$$

and

$$\hat{\gamma}_{2m+1}(2m + 1) = (-\mathrm{i})^m \sigma_3 \circ J(2m). \tag{11.14}$$

(1) Show

$$\hat{\gamma}_j(2m + 1)\hat{\gamma}_k(2m + 1) + \hat{\gamma}_k(2m + 1)\hat{\gamma}_j(2m + 1) = 2\delta_{jk}I.$$

(2) Show

$$J(2m+1) = \mathrm{i}^m \sigma_3 \circ I.$$

(Parts (1) and (2) show that the set $\hat{\gamma}_k(2m+1)$ generates the universal Clifford algebra $\boldsymbol{C}_{2m+1,0}$.)

(3) Use the result of part (2) to show that

$$\tfrac{1}{2}[I + (-\mathrm{i})^m J(2m+1)]\hat{\gamma}_k(2m+1) = \left[\begin{array}{c:c} \hat{\gamma}_k(2m) & 0 \\ \hdashline 0 & 0 \end{array}\right]$$

and

$$\tfrac{1}{2}[I + (-\mathrm{i})^m J(2m+1)]\hat{\gamma}_k(2m+1) = \left[\begin{array}{c:c} 0 & 0 \\ \hdashline 0 & \hat{\gamma}_k(2m) \end{array}\right]$$

(This result should convince you that $\boldsymbol{C}_{2m+1,0}$ is isomorphic to the algebra of matrices of the form $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$ where A and B are arbitrary complex matrices of size $2^m \times 2^m$. This algebra is known as the *direct sum* of $\boldsymbol{C}(2^m)$ with itself. This particular direct sum is designated by $\boldsymbol{C}(2^m) \oplus \boldsymbol{C}(2^m)$ or more simply by ${}^2\boldsymbol{C}(2^m)$.

(4) Adjust the definitions of Eqs. (11.13) and (11.14) so that the resulting Dirac matrices are suitable for $\boldsymbol{C}_{p,q}$. Then convince yourself that the basic result of part (3) is valid for $\boldsymbol{C}_{p,q}$, that is $\boldsymbol{C}_{p,q}$ is isomorphic to ${}^2\boldsymbol{C}(2^m)$, where $p + q = 2m + 1$.

11.2 The Classification of all Real Finite Dimensional Clifford Algebras

In the last section, it was shown that the universal Clifford algebra $\boldsymbol{C}_{p,q}$ is isomorphic to $\boldsymbol{C}(2^m)$ if $p + q = 2m$ and isomorphic to ${}^2\boldsymbol{C}(2^m)$ if $p + q = 2m + 1$. The classification of the real Clifford algebras is more intricate. William Kingdom Clifford made some headway on this task (1882). Although he recognized that it was possible for $\hat{\gamma}_{123\ldots n}$ to be a scalar multiple of I, it appears that he failed to recognize that not all Clifford algebras are uniquely determined by the requirement that $\hat{\gamma}_j\hat{\gamma}_k + \hat{\gamma}_k\hat{\gamma}_j = 2n_{jk}I$. It is quite possible that this prevented him from making more progress on this matter during his lifetime. Since his death, this problem has been solved.

In a paper dedicated to Arnold Shapiro, Michael F. Atiyah and Raoul H. Bott published a paper in which all real Clifford algebras of types $\boldsymbol{R}_{p,0}$ and $\boldsymbol{R}_{0,q}$ are classified (1964). Their methods are extended to all Clifford algebras in *Topological Geometry* (Porteus 1981, pp. 240–251). The results are most neatly summarized in a paper by Pertti Lounesto (1980).

To present the methods of Bott and Atiyah, it is useful to introduce a few formal definitions and some additional notation. An algebra $\boldsymbol{A}$ is said to be *generated* by a subset S if every member of $\boldsymbol{A}$ can be expressed as a linear combination of finite products of the elements of S. (Such expressions may not be unique.) We will write

$$\boldsymbol{A} = [a_1, a_2, \ldots, a_n]$$

if A is generated by the set

$$S = \{a_1, a_2, \ldots, a_n\}.$$

A set of generators is by no means unique. Thus we can write

$$\boldsymbol{R}_{p,q} = [\hat{\gamma}_1, \hat{\gamma}_2, \hat{\gamma}_3, \ldots, \hat{\gamma}_n]$$

or alternatively

$$\boldsymbol{R}_{p,q} = [\hat{\gamma}_1, \hat{\gamma}_2, \hat{\gamma}_{312}, \hat{\gamma}_{412}, \hat{\gamma}_{512}, \ldots, \hat{\gamma}_{n12}].$$

Another key definition is that of *direct product* or *tensor product*. An algebra $\boldsymbol{A}$ is said to be the direct product or tensor product of two subalgebras $\boldsymbol{B}$ and $\boldsymbol{D}$ if

(1) $\boldsymbol{A}$ is generated by the elements of $\boldsymbol{B}$ and $\boldsymbol{D}$.
(2) $\dim \boldsymbol{A} = \dim \boldsymbol{B} \dim \boldsymbol{D}$.
(3) $\forall b \in \boldsymbol{B}$ and $d \in \boldsymbol{D}$, $bd = db$.

If these three conditions are satisfied, then one writes $\boldsymbol{A} = \boldsymbol{B} \otimes \boldsymbol{D}$. If the field of scalars is unclear from the context, this is incorporated into the notation. For example, if the field of scalars is the field of real numbers $\mathbb{R}$, one writes

$$\boldsymbol{A} = \boldsymbol{B} \otimes_{\mathbb{R}} \boldsymbol{D}.$$

In the context of matrices, the term Kronecker product is frequently used as a synonym for direct or tensor product. This may seem strange since the Kronecker product is not commutative. For example $\sigma_1 \circ \sigma_3 \neq \sigma_3 \circ \sigma_1$. However,

$$(\sigma_1 \circ I)(I \circ \sigma_3) = (I \circ \sigma_3)(\sigma_1 \circ I).$$

For this reason, it will become apparent that it is only a slight (if any) abuse of terminology to use the term Kronecker product as a synonym for direct product.

We are now in a position to apply the methods of Bott and Atiyah to

classify all real Clifford algebras. We first note that

$$\begin{aligned} \boldsymbol{R}_{p,q} &= [\hat{\gamma}_1, \hat{\gamma}_2, \hat{\gamma}_{312}, \hat{\gamma}_{412}, \hat{\gamma}_{512}, \ldots, \hat{\gamma}_{n12}] \\ &= [\hat{\gamma}_1, \hat{\gamma}_2] \otimes [\hat{\gamma}_{312}, \hat{\gamma}_{412}, \hat{\gamma}_{512}, \ldots, \hat{\gamma}_{n12}]. \end{aligned} \tag{11.15}$$

If $(\hat{\gamma}_1)^2 = (\hat{\gamma}_2)^2 = +I$, then

$$(\hat{\gamma}_{k12})^2 = -\hat{\gamma}_{k12}\hat{\gamma}_{21k} = -(\hat{\gamma}_k)^2.$$

If the $\hat{\gamma}_{k12}$'s are treated as the Dirac matrices of a Clifford algebra and $p \geqslant 2$, then

$$[\hat{\gamma}_{312}, \gamma_{412}, \gamma_{512}, \ldots, \gamma_{n12}] = \boldsymbol{R}_{q,p-2}.$$

Thus in this case, Eq. (11.15) becomes

$$\boldsymbol{R}_{p,q} = \boldsymbol{R}_{2,0} \otimes \boldsymbol{R}_{q,p-2}. \tag{11.16}$$

If $p \geqslant 1$ and $q \geqslant 1$, we also have

$$\begin{aligned} R_{p,q} &= [\hat{\gamma}_1, \hat{\gamma}_n, \hat{\gamma}_{21n}, \hat{\gamma}_{31m}, \hat{\gamma}_{41n}, \ldots, \hat{\gamma}_{(n-1)1n}] \\ &= [\hat{\gamma}_1, \hat{\gamma}_n] \otimes [\hat{\gamma}_{21n}, \hat{\gamma}_{31n}, \hat{\gamma}_{41n}, \ldots, \hat{\gamma}_{(n-1)1n}] \end{aligned}$$

This time

$$(\hat{\gamma}_{k1n})^2 = -\hat{\gamma}_{k1n}\hat{\gamma}_{n1k} = +(\hat{\gamma}_k)^2,$$

so

$$\boldsymbol{R}_{p,q} = \boldsymbol{R}_{1,1} \otimes \boldsymbol{R}_{p-1,q-1}. \tag{11.17}$$

In a similar fashion if $q \geqslant 2$, then we can show

$$\boldsymbol{R}_{p,q} = \boldsymbol{R}_{0,2} \otimes \boldsymbol{R}_{q-2,p}. \tag{11.18}$$

From Eqs. (11.1) and (11.2), we know that $\boldsymbol{R}_{2,0} = \boldsymbol{R}_{1,1} = \boldsymbol{R}(2)$ and $\boldsymbol{R}_{0,2} = \boldsymbol{H}$. Thus, we already see that if $p + q$ is even then $\boldsymbol{R}_{p,q}$ can be decomposed into a multiple direct product of $\boldsymbol{R}(2)$'s and $\boldsymbol{H}$'s. The labor of classifying all Clifford algebras is substantially reduced by the consequences of the following theorem.

Theorem 11.3. $\boldsymbol{R}_{p+4k,q-4k} = \boldsymbol{R}_{p,q}$ where k is any positive or negative integer consistent with the requirement that $p + 4k \geqslant 0$ and $q - 4k \geqslant 0$.

Proof. To prove this theorem, it is only necessary to show that $\boldsymbol{R}_{p+4,q-4} =$

$\boldsymbol{R}_{p,q}$. Applying Eqs. (11.16) and (11.18), one gets

$$\boldsymbol{R}_{p+4,q-4} = \boldsymbol{R}_{2,0} \otimes \boldsymbol{R}_{q-4,p+2} = \boldsymbol{R}_{2,0} \otimes \boldsymbol{R}_{0,2} \otimes \boldsymbol{R}_{p,q-4}. \tag{11.19}$$

Applying the same two equations in reverse order, one also gets

$$\boldsymbol{R}_{p,q} = \boldsymbol{R}_{0,2} \otimes \boldsymbol{R}_{q-2,p} = \boldsymbol{R}_{0,2} \otimes \boldsymbol{R}_{2,0} \otimes \boldsymbol{R}_{p,q-4}. \tag{11.20}$$

Our desired result is an immediate consequence of Eqs. (11.19) and (11.20). ■

This last theorem implies that for any value of $p + q$, we need only compute $\boldsymbol{R}_{m+k,m-k}$ for $k = -1, 0, 1$, and 2. However, even this task is cut almost in half by the following theorem.

Theorem 11.4. For a fixed value of $p + q$, $\boldsymbol{R}_{p,q}$ is an even function of $p - q - 1$; that is

$$\boldsymbol{R}_{(p+q+1)/2+(p-q-1)/2,(p+q-1)/2-(p-q-1)/2} = \boldsymbol{R}_{(p+q+1)/2-(p-q-1)/2,(p+q-1)/2+(p-q-1)/2}$$

or

$$\boldsymbol{R}_{p,q} = \boldsymbol{R}_{q+1,p-1}.$$

Proof. Following the approach of Porteus (1981, p. 248), we note that

$$\boldsymbol{R}_{p,q} = [\hat{\gamma}_1, \hat{\gamma}_{21}, \hat{\gamma}_{31}, \hat{\gamma}_{41}, \ldots, \hat{\gamma}_{n1}].$$

If $(\hat{\gamma}_1)^2 = +I$, then $(\hat{\gamma}_{k1})^2 = -\hat{\gamma}_{k1}\hat{\gamma}_{1k} = -(\hat{\gamma}_k)^2$. Thus if $\hat{\gamma}_1, \hat{\gamma}_{21}, \hat{\gamma}_{31}, \ldots, \hat{\gamma}_{n1}$ are treated as Dirac matrices, it is clear that they generate $\boldsymbol{R}_{q+1,p-1}$. ■

From this last theorem, $\boldsymbol{R}_{m+2,m-2} = \boldsymbol{R}_{m-1,m+1}$ and $\boldsymbol{R}_{m+1,m-1} = \boldsymbol{R}_{m,m}$. Thus if $p + q$ is even we need only compute $\boldsymbol{R}_{m+k,m-k}$ for $k = 0$ and -1. The simpler one is $\boldsymbol{R}_{m,m}$. By repeated application of Eq. (11.17), we have

$$\boldsymbol{R}_{m,m} = \boldsymbol{R}_{1,1} \otimes \boldsymbol{R}_{1,1} \otimes \cdots \otimes \boldsymbol{R}_{1,1}$$

or

$$\boldsymbol{R}_{m,m} = \boldsymbol{R}(2) \otimes \boldsymbol{R}(2) \otimes \cdots \otimes \boldsymbol{R}(2) \tag{11.21}$$

where each factor appears m times.

To simplify Eq. (11.21), it will be shown that

$$\boldsymbol{R}(r) \otimes \boldsymbol{R}(s) = \boldsymbol{R}(rs). \tag{11.22}$$

To prove Eq. (11.22), let $E_{ij}(r)$ denote the $r \times r$ matrix with 1 in the ith row and jth column and 0 in all other positions. Also let I_r denote the $r \times r$ identity matrix. Then

$$\begin{aligned} \boldsymbol{R}(r) \otimes \boldsymbol{R}(s) &= [E_{11}(r) \circ I_s, E_{12}(r) \circ I_s, \ldots, E_{ij}(r) \circ I_s, \ldots] \\ &\quad \otimes [I_r \circ E_{11}(s), \ldots, I_r \circ E_{mk}(s), \ldots] \\ &= [\ldots, E_{ij}(r) \circ E_{mk}(s), \ldots] \\ &= \boldsymbol{R}(rs). \end{aligned}$$

From Eqs. (11.21) and (11.22), we now have

$$\boldsymbol{R}_{m,m} = \boldsymbol{R}(2^m). \tag{11.23}$$

To compute $\boldsymbol{R}_{m-1,m+1}$, we note that from Eq. (11.18), we have

$$\boldsymbol{R}_{m-1,m+1} = \boldsymbol{R}_{0,2} \otimes \boldsymbol{R}_{m-1,m-1} = \boldsymbol{H} \otimes \boldsymbol{R}(2^{m-1}). \tag{11.24}$$

In general

$$\boldsymbol{H} \otimes \boldsymbol{R}(r) = [1 \circ I_r, \boldsymbol{i} \circ I_r, \boldsymbol{j} \circ I_r, \boldsymbol{k} \circ I_r] \otimes [\ldots, 1 \circ E_{ij}(r), \ldots] = \boldsymbol{H}(r)$$

which is the algebra of $r \times r$ matrices whose components are arbitrary quaternions.

From Eq. (11.24), we now have

$$\boldsymbol{R}_{m-1,m+1} = \boldsymbol{H}(2^{m-1}). \tag{11.25}$$

Let us now turn to the real universal Clifford algebras for the case that $p + q$ is odd. From Theorem 11.3, it is only necessary to compute $\boldsymbol{R}_{m+1+k,m-k}$ for $k = -1, 0, 1,$ and 2. From Theorem 11.4, $\boldsymbol{R}_{m+1+k,m-k} = \boldsymbol{R}_{m+1-k,m+k}$. Thus we need only compute $\boldsymbol{R}_{m+1+k,m-k}$ for $k = 0, 1,$ and 2.

We note that

$$\begin{aligned} \boldsymbol{R}_{m+1+k,m-k} &= [\hat{\gamma}_1, \hat{\gamma}_2, \ldots, \hat{\gamma}_{2m+1}] \\ &= [\hat{\gamma}_2, \hat{\gamma}_3, \ldots, \hat{\gamma}_{2m+1}, J] \\ &= [J] \otimes [\hat{\gamma}_2, \hat{\gamma}_3, \ldots, \hat{\gamma}_{2m+1}]. \end{aligned} \tag{11.26}$$

Because of the values of k which are under consideration, we may require that $(\hat{\gamma}_1)^2 = +I$. In that case Eq. (11.26) becomes

$$\boldsymbol{R}_{m+1+k,m-k} = [J] \otimes \boldsymbol{R}_{m+k,m-k}. \tag{11.27}$$

From Eq. (11.6),

$$\begin{aligned} J^2 &= (-1)^{(2m+1)2m/2}(-1)^q I = (-1)^{2m^2}(-1)^m(-1)^q I = (-1)^m(-1)^{-q} I \\ &= (-1)^k I. \end{aligned}$$

For both odd and even values of k, $[J]$ is a 2-dimensional algebra. For even values of k, $J^2 = +I$ and we can represent J by the matrix

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

In that case $[J]$ may be regarded as the algebra of 2×2 matrices of the form $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ whre a and b are real numbers. This algebra is sometimes denoted as the direct sum $\boldsymbol{R} \oplus \boldsymbol{R}$ or more simply by the symbol ${}^2\boldsymbol{R}$.

For odd values of k, $J^2 = -I$ and we can represent J by i. In this case $[J]$ may be identified as the algebra of complex numbers $\mathbb{C}$. In this context, $\mathbb{C}$ is a 2-dimensional algebra since 1 and i are linearly independent over the field of real numbers.

Now we are in a position to complete the computation of the real universal Clifford algebras for odd values of $p + q$. By successively substituting $k = 0$, 1, and 2 into Eq. (11.27) and using Theorem 11.4 where needed, we have

$$\boldsymbol{R}_{m+1,m} = {}^2\boldsymbol{R} \otimes \boldsymbol{R}_{m,m},$$

$$\boldsymbol{R}_{m+2,m-1} = \boldsymbol{C} \otimes \boldsymbol{R}_{m+1,m-1} = \boldsymbol{C} \otimes \boldsymbol{R}_{m,m},$$

and

$$\boldsymbol{R}_{m+3,m-2} = {}^2\boldsymbol{R} \otimes \boldsymbol{R}_{m+2,m-2} = {}^2\boldsymbol{R} \otimes \boldsymbol{R}_{m-1,m+1}.$$

Using Eqs. (11.23) and (11.25), these equations become

$$\boldsymbol{R}_{m+1,m} = {}^2\boldsymbol{R} \otimes \boldsymbol{R}(2^m), \tag{11.28}$$

$$\boldsymbol{R}_{m+2,m-1} = \boldsymbol{C} \otimes \boldsymbol{R}(2^m), \tag{11.29}$$

and

$$\boldsymbol{R}_{m+3,m-2} = {}^2\boldsymbol{R} \otimes \boldsymbol{H}(2^{m-1}). \tag{11.30}$$

Now

$$
{}^2\boldsymbol{R} \otimes \boldsymbol{R}(r) = \left[\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \circ I_r, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \circ I_r \right] \otimes [\ldots, I_2 \circ E_{ij}(r), \ldots]
$$

$$
= \left[\cdots \left[\begin{array}{c:c} E_{ij}(r) & 0 \\ \hdashline 0 & 0 \end{array} \right], \left[\begin{array}{c:c} 0 & 0 \\ \hdashline 0 & E_{ij}(r) \end{array} \right], \cdots \right]
$$

which equals the algebra of real matrices of the form

$$
\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}
$$

where A and B are arbitrary real matrices of size $r \times r$. This algebra is generally designated by the direct sum $\boldsymbol{R}(r) \oplus \boldsymbol{R}(r)$ or more simply by the symbol ${}^2\boldsymbol{R}(r)$.

Applying similar arguments to the tensor products that appear in Eqs. (11.29) and (11.30), we have

$$
\boldsymbol{R}_{m+1,m} = {}^2\boldsymbol{R}(2^m), \tag{11.31}
$$

$$
\boldsymbol{R}_{m+2,m-1} = \boldsymbol{C}(2^m), \tag{11.32}
$$

and

$$
\boldsymbol{R}_{m+3,m-2} = {}^2\boldsymbol{H}(2^{m-1}). \tag{11.33}
$$

All universal real Clifford algebras have now been computed. The results are summarized in Table 11.1.

Using the results obtained for the real universal Clifford algebras, it is not too difficult to classify the real nonuniversal Clifford algebras. (See Problem 11.4.) The classification of the real nonuniversal Clifford algebras is summarized in Table 11.2.

Problem 11.4. Use the results of Problem 11.1 to construct Table 11.2 from Table 11.1.

Problem 11.5. Show that a nonuniversal complex Clifford algebra can be constructed for any signature (p, q) such that $p + q = 2m + 1$. Show that any such Clifford algebra is isomorphic to $\boldsymbol{C}(2^m)$.

Table 11.1. Classificaion of all real universal Clifford algebras: $\boldsymbol{R}_{p,q}$.

$\lvert p - q - 1 \pm 8k \rvert$	0	1	2	3	4
$p + q = 2m$		$\boldsymbol{R}(2^m)$		$\boldsymbol{H}(2^{m-1})$	
$p + q = 2m + 1$	${}^2\boldsymbol{R}(2^m)$		$\boldsymbol{C}(2^m)$		${}^2\boldsymbol{H}(2^{m-1})$

Table 11.2. The classification of all real nonuniversal Clifford algebras: $\bar{R}_{p,q}$.

$\lvert p-q-1 \pm 8k \rvert =$	0	4
$p+q=2m+1$	$R(2^m)$	$H(2^{m-1})$

11.3 The Classification of all c-Unitary Groups

Among the Clifford numbers belonging to any complex Clifford algebra $C_{p,q}$ are the c-unitary Clifford numbers. As you may recall, a Clifford number U is c-unitary if $UU^\dagger = I$ where $U^\dagger$ is the complex reverse of U. The set of c-unitary Clifford numbers belonging to $C_{p,q}$ form a group which is hereby designated as the *c-unitary group* $\mathrm{CU}(p, q)$.

In the discussion of the Clifford number solutions to Dirac's equation for the electron, it was noted that expectation values are invariant when a solution is multiplied by a c-unitary Clifford number. Thus c-unitary groups may form the basis for some useful gauge theory. With this thought in mind, this section is devoted to the task of classifying these groups.

Perhaps the easiest c-unitary groups to deal with are those of type $\mathrm{CU}(2m, 0)$. In the first section of this chapter near Eq. (11.10), a scheme was presented for constructing matrix representations for Clifford algebras. In that scheme

$$\hat{\gamma}_k(2m) = c_k m_{k_1} \circ m_{k_2} \circ \cdots \circ m_{k_m} \tag{11.34}$$

where $m_{k_j} = I(2)$, σ_1, σ_2, or σ_3 and $c_k = 1, -1, \mathrm{i}$, or $-\mathrm{i}$.

For $C_{2m,0}$, $(\hat{\gamma}_k)^2 = +I$ for each value of k. This means that c_k is real for each value of k. Furthermore the Kronecker product of Hermite matrices is Hermite. Since the Pauli matrices are Hermite, it follows that for $C_{2m,0}$ the matrix representation of Eq. (11.34) results in Hermite matrices for all of the Dirac matrices. In this situation it is a trivial task to show that taking the complex reversal of a Clifford number is synonymous with taking the complex transpose of its matrix representation. In particular, if

$$\mathscr{A} = \sum A^{i_1 i_2 \ldots i_k} \hat{\gamma}_{i_1} \hat{\gamma}_{i_2} \hat{\gamma}_{i_3} \cdots \hat{\gamma}_{i_k}$$

then

$$\begin{aligned}\mathscr{A}^* &= \sum \bar{A}^{i_1 i_2 \ldots i_k} \hat{\gamma}^*_{i_k} \hat{\gamma}^*_{i_{k-1}} \cdots \hat{\gamma}^*_{i_2} \hat{\gamma}^*_{i_1} \\ &= \sum \bar{A}^{i_1 i_2 \ldots i_k} \hat{\gamma}_{i_k} \cdots \hat{\gamma}_{i_2} \hat{\gamma}_{i_1} = \mathscr{A}^\dagger.\end{aligned}$$

Since taking the complex reversal is synonymous with taking the complex transpose of its matrix representation, it is obvious that the group of c-unitary matrices $\mathrm{CU}(2m, 0)$ may be identified with the group $\mathrm{U}(2^m)$--- that is the group of unitary matrices of size $2^m \times 2^m$.

For Clifford algebras of type $C_{2m+1,0}$, the situation is only slightly more

complicated. Here again computing the complex reversal is synonymous with computing the complex transpose of the matrix representation presented in Section 11.2. However, here the group of unitary matrices consists of those to be found in the algebra $C(2^m) \oplus C(2^m)$. This group is denoted by $\mathrm{U}(2^m) \oplus \mathrm{U}(2^m)$. Members of this group have the form

$$\begin{bmatrix} U_1 & 0 \\ 0 & U_2 \end{bmatrix}$$

where U_1 and U_2 are unitary matrices of size $2^m \times 2^m$.

For $C_{p,q}$ where $q \neq 0$ and $p + q = 2m$, the situation is substantially more involved. In this case Eq. (11.34) remains valid but $c_k = \pm 1$ only for $k = 1, 2, \ldots, p$. For $k = p + 1, p + 2, \ldots, p + q$, $c_k = \pm \mathrm{i}$. This means that

$$\hat{\gamma}_k^*(2m) = \hat{\gamma}_k(2m) \quad \text{for } k = 1, 2, \ldots, p$$

and

$$\hat{\gamma}_k^*(2m) = -\hat{\gamma}_k(2m) \quad \text{for } k = p + 1, p + 2, \ldots, p + q.$$

To deal with this case it is useul to introduce a matrix M which commutes with the first p Dirac matrices and anticommutes with the last q Dirac matrices.

If p is odd, let

$$M = \hat{\gamma}_{12\ldots p}. \tag{11.35}$$

If p is even, then q is also even. In that case let

$$M = \hat{\gamma}_{p+1}\hat{\gamma}_{p+2} \cdots \hat{\gamma}_{p+q} \qquad (\text{note:} \quad q \neq 0). \tag{11.36}$$

We now have

$$\hat{\gamma}_k(2m)M = \hat{\gamma}_k^\dagger(2m)M = M\hat{\gamma}_k^*(2m) \quad \text{for } k = 1, 2, \ldots, 2m.$$

The reader should not have trouble generalizing the second equality to any Clifford number, that is

$$\mathscr{A}^\dagger M = M\mathscr{A}^*.$$

This means that

$$UU^\dagger M = UMU^*.$$

Thus

$$U \text{ is } c\text{-unitary} \Leftrightarrow UMU^* = M. \tag{11.37}$$

From the Kronecker product construction of M, it is clear that either M or iM is Hermite. If M is not Hermite, multiply M by i and relabel the resulting matrix by M. This will not modify Eq. (11.37). However, the adjusted M will have the property that $M^2 = +I$. This means that the eigenvalues of M will be ± 1. From the Kronecker product construction of M, it is also clear that the trace of M is 0. This means that there exists a unitary matrix P such that

$$PMP^* = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}$$

where I is the identity matrix of size $2^{m-1} \times 2^{m-1}$.

If we replace each of the Dirac matrices $\hat{\gamma}_k$ by $P\hat{\gamma}_k P^*$, the Clifford algebra remains the same but Eq. (11.37) becomes

$$U\begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}U^* = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}.$$

The group of matrices that satisfies this relation is denoted by $\mathrm{U}(2^{m-1}, 2^{m-1})$. Thus if $q \neq 0$ and $p + q = 2m$, then

$$\mathrm{CU}(p, q) = \mathrm{U}(2^{m-1}, 2^{m-1}).$$

For the case where $p + q = 2m + 1$, the situation is even more involved. However, it is still possible to construct a matrix M that commutes with the first p Dirac matrices and anticommutes with the last q Dirac matrices. If p is odd, let

$$M = \hat{\gamma}_{12\ldots p}(2m + 1).$$

To get a useful matrix representation for M, let us construct $\boldsymbol{C}_{p,q}$ from $\boldsymbol{C}_{p,q-1}$, that is, let

$$\hat{\gamma}_k(2m + 1) = \sigma_3 \circ \hat{\gamma}_k(2m) \quad \text{for } k = 1, 2, \ldots, 2m$$

and

$$\hat{\gamma}_{2m+1}(2m + 1) = c\sigma_3 \circ J(2m)$$

where

$$c \text{ is chosen so that } [\hat{\gamma}_{2m+1}(2m + 1)]^2 = -I.$$

Since p is odd, it is clear that

$$M(2m + 1) = \sigma_3 \circ \hat{\gamma}_{12\ldots p}(2m) = \sigma_3 \circ M(2m).$$

After the same adjustments used before, we can replace $M(2m)$ by a diagonal matrix of size $2^m \times 2^m$ with an equal number of $+1$'s and -1's on the diagonal.

Under this circumstance, the group of matrices that satisfies the equation $UMU^* = M$ is $\mathrm{U}(2^{m-1}, 2^{m-1}) \oplus \mathrm{U}(2^{m-1}, 2^{m-1})$.

Members of this group are matrices of the form

$$\begin{bmatrix} U_1 & 0 \\ 0 & U_2 \end{bmatrix}$$

where U_1 and U_2 are members of the group $\mathrm{U}(2^{m-1}, 2^{m-1})$.

Finally, we have the case where p is even and q is odd. To deal with this case, it is useful to identify the members of $\boldsymbol{C}_{p,q}$ with linear combinations of even order forms in $\boldsymbol{C}_{p+1,q}$. In particular let

$$\hat{\gamma}_k(2m+1) = \mathrm{i}\hat{\gamma}_1(2m+2)\hat{\gamma}_{k+1}(2m+2) \quad \text{for } k = 1, 2, \ldots, 2m+1. \tag{11.38}$$

It is not difficult to show that elements on the right-hand side of Eq. (11.38) generate forms of even order associated with $\boldsymbol{C}_{p+1,q}$. Also the mapping preserves both the operation of complex reversal and complex transpose. This means that if we find a matrix M is $\boldsymbol{C}_{p+1,q}$ such that

$$\mathscr{A}^\dagger M = M\mathscr{A}^*$$

for an arbitrary Clifford number $\mathscr{A}$ in $\boldsymbol{C}_{p+1,q}$, the same relation will hold when the $\mathscr{A}$'s are restricted to forms of even order in $\boldsymbol{C}_{p+1,q}$. Since these even order forms in $\boldsymbol{C}_{p+1,q}$ correspond to the forms of arbitrary order in $\boldsymbol{C}_{p,q}$, this M will serve our purpose.

What makes this situation somewhat different from those previously considered is that this time the matrix M does not belong to the matrix representation of $\boldsymbol{C}_{p,q}$. This is because it is a form of odd order in $\boldsymbol{C}_{p+1,q}$. In particular, let

$$M = \hat{\gamma}_1(2m+2)\hat{\gamma}_2(2m+2)\ldots\hat{\gamma}_{p+1}(2m+2). \tag{11.39}$$

Examining the construction scheme outlined near Eq. (11.10), it becomes obvious that

$$M = (\sigma_1)^{p+1} \circ W = \sigma_1 \circ W. \tag{11.40}$$

(Since q is odd, $q \geqslant 1$ so $\hat{\gamma}_{2m+2}(2m+2)$, which equals $\sigma_2 \circ I$, does not enter into the product on the right-hand side of Eq. (11.39).) The matrix W is an invertible matrix of size $2^{2m} \times 2^{2m}$. Now with our representation, any member of $\boldsymbol{C}_{p,q}$ must be of the form

$$\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}.$$

From Eq. (11.40), the matrix M must be of the form

$$\begin{bmatrix} 0 & W \\ W & 0 \end{bmatrix}.$$

Thus the equation, "$UMU^* = M$", becomes

$$\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}\begin{bmatrix} 0 & W \\ W & 0 \end{bmatrix}\begin{bmatrix} A^* & 0 \\ 0 & B^* \end{bmatrix} = \begin{bmatrix} 0 & W \\ W & 0 \end{bmatrix}.$$

Multiplying this out, one gets

$$AWB^* = W \tag{11.41}$$

and

$$BWA^* = W. \tag{11.42}$$

From Eq. (11.41), $B^* = W^{-1}A^{-1}W$ and

$$B = W^*(A^*)^{-1}(W^*)^{-1}. \tag{11.43}$$

From Eq. (11.42), we have

$$B = W(A^*)^{-1}W^{-1}. \tag{11.44}$$

Equations (11.43) and (11.44) agree without imposing any restriction on $(A^*)^{-1}$ since from the Kronecker product construction of W, we know that $W^* = \pm W$. The only restriction on A is that $(A^*)^{-1}$ must exist. Thus any member of $\mathrm{CU}(p, q)$ must be of the form

$$U = \begin{bmatrix} A & 0 \\ 0 & W(A^*)^{-1}W^{-1} \end{bmatrix} \tag{11.45}$$

where A is any complex invertible matrix of size $2^m \times 2^m$. There is an obvious correspondence between matrices on the right-hand side of Eq. (11.45) and matrices of the form $[A]$. This correspondence is actually an isomorphism. Thus the group $\mathrm{CU}(2k, 2m - 2k + 1)$ is isomorphic to the group of invertible matrices of size $2^m \times 2^m$ with complex components. This group is known as the *general linear group* of $2^m \times 2^m$ matrices over the complex numbers and is denoted by $\mathrm{GL}(2^m, \mathbb{C})$.

To summarize the results of this section, $\mathrm{CU}(p, q)$ denotes the group of c-unitary elements of $C_{p,q}$ and

$$\mathrm{CU}(2m, 0) = \mathrm{U}(2^m), \tag{11.46}$$

$$\mathrm{CU}(2m+1, 0) = \mathrm{U}(2^m) \oplus \mathrm{U}(2^m), \tag{11.47}$$

$$\mathrm{CU}(2m-k, k) = \mathrm{U}(2^{m-1}, 2^{m-1}) \quad \text{for } k \geqslant 1, \tag{11.48}$$

$$\mathrm{CU}(2k+1, 2m-2k) = \mathrm{U}(2^{m-1}, 2^{m-1}) \oplus \mathrm{U}(2^{m-1}, 2^{m-1}) \quad \text{for } m > k, \tag{11.49}$$

and

$$\mathrm{CU}(2k, 2m+1-2k) = \mathrm{GL}(2^m, \mathbb{C}). \tag{11.50}$$

Problem 11.6. Show that for the mapping of Eq. (11.38),

$$(\hat{\gamma}_k(2m+1))^2 = I \quad \text{for } k = 1, 2, \ldots, p$$

and

$$(\hat{\gamma}_k(2m+1))^2 = -I \quad \text{for } k = p+1, p+2, \ldots, 2m+1.$$

Problem 11.7. Consider the mapping

$$\hat{\gamma}_k(2m+1) = \mathrm{i}\hat{\gamma}_{2m+2-k}(2m+2)\hat{\gamma}_{2m+2}(2m+2) \quad \text{for } k = 1, 2, \ldots, 2m+1.$$

Show that this mapping generates an isomorphism between the forms of arbitrary order in $\boldsymbol{C}_{p,q}$ with the even order forms in $\boldsymbol{C}_{q+1,p}$ where $p+q = 2m+1$.

APPENDIX

A.1 The Product Decomposition of Restricted Lorentz Operators and Related Operators

This section is devoted to some particular factor decompositions of restricted Lorentz transformations and related operators. However, the computational techniques introduced in this appendix may have equal importance.

The reader may first ask, what is a restricted Lorentz transformation? In Chapter 2, boost operators and rotation operators for Minkowski 4-space were discussed. One may ask, what happens when one takes a product of several boosts and rotations? One knows that the composition of any number of rotations can be expressed as a single rotation. However, it is not difficult to show that the composition of two boosts is not another boost unless the directions of the two boosts are the same.

Both a boost operator and a rotation operator is a linear combination of p-vectors of even order. Thus any product of boosts and rotations must have this same property. In addition for both a boost and a rotation operator, one can compute the inverse operator by reversing the order of the Dirac matrices. It is not difficult to show that this property must also hold for an arbitrary product of boosts and rotations.

Suppose $\mathscr{A}^\dagger$ designates the Clifford number obtained from $\mathscr{A}$ by reversing the order of the Dirac matrices in $\mathscr{A}$. For example, if

$$\mathscr{A} = aI + b\hat{\gamma}_{12} + c\hat{\gamma}_{20} + d\hat{\gamma}_{123} + e\hat{\gamma}_{0123},$$

then

$$\begin{aligned}\mathscr{A}^\dagger &= aI + b\hat{\gamma}_{21} + c\hat{\gamma}_{02} + d\hat{\gamma}_{321} + e\hat{\gamma}_{3210}\\ &= aI - b\hat{\gamma}_{12} - c\hat{\gamma}_{20} - d\hat{\gamma}_{123} + e\hat{\gamma}_{0123}.\end{aligned}$$

$\mathscr{A}^\dagger$ is called the *Clifford reverse* of $\mathscr{A}$. We note that

$$\begin{aligned}[(\hat{\gamma}_{j_1}\hat{\gamma}_{j_2}\cdots\hat{\gamma}_{j_m})(\hat{\gamma}_{k_1}\hat{\gamma}_{k_2}\cdots\hat{\gamma}_{k_n})]^\dagger &= (\hat{\gamma}_{k_n}\cdots\hat{\gamma}_{k_2}\hat{\gamma}_{k_1})(\hat{\gamma}_{j_m}\cdots\hat{\gamma}_{j_2}\hat{\gamma}_{j_1})\\ &= (\hat{\gamma}_{k_1}\hat{\gamma}_{k_1}\cdots\hat{\gamma}_{k_n})^\dagger(\hat{\gamma}_{j_1}\hat{\gamma}_{j_2}\cdots\hat{\gamma}_{j_m})^\dagger.\end{aligned}$$

The reader should not have difficulty generalizing this first to a product of two Clifford numbers and then to a product of any finite product of Clifford numbers. That is

$$(\mathscr{A}\mathscr{B})^\dagger = \mathscr{B}^\dagger\mathscr{A}^\dagger \qquad \text{and} \qquad (\mathscr{A}_1\mathscr{A}_2\ldots\mathscr{A}_n)^\dagger = \mathscr{A}_n^\dagger\ldots\mathscr{A}_2^\dagger\mathscr{A}_1^\dagger.$$

If $\mathscr{A}_k^\dagger = \mathscr{A}_k^{-1}$ for $k = 1, 2, \ldots, n$, then

$$\begin{aligned}(\mathscr{A}_1\mathscr{A}_2\ldots\mathscr{A}_k)^\dagger &= \mathscr{A}_k^\dagger\ldots\mathscr{A}_2^\dagger\mathscr{A}_1^\dagger\\ &= \mathscr{A}_k^{-1}\ldots\mathscr{A}_2^{-1}\mathscr{A}_1^{-1}.\\ &= (\mathscr{A}_1\mathscr{A}_2\ldots\mathscr{A}_k)^{-1}.\end{aligned}$$

Thus, if $\mathscr{L}$ is an arbitrary product of boosts and rotations then $\mathscr{L}^\dagger = \mathscr{L}^{-1}$.

It will be shown in this section that if $\mathscr{L}$ is an operator composed of a linear combination of even order vectors and $\mathscr{L}^\dagger = \mathscr{L}^{-1}$ then $\mathscr{L}$ can be factored into a product of a single boost and a single rotation in a unique manner. Such an operator $\mathscr{L}$ is a *restricted Lorentz transformation.* Such transformations are said to be restricted because as a set they do not include reflections or translations.

In this section, I will discuss not only the factorization of restricted Lorentz operators mentioned above but other factor decompositions not only for restricted Lorentz operators but also for slightly more general operators which must be dealt with in discussing interpretations of Dirac's equation for the electron.

To deal with the computations necessary for any of these factor decompositions, I have found it useful to use a theorem due to Joseph H. M. Wedderburn (Albert 1939, p. 39) on the structure of algebras. When this theorem is applied to the Clifford algebra of Minkowski 4-space, we find that it implies that this particular algebra can be decomposed into a direct product of an algebra generated by quaternions and the algebra of real 2×2 matrices. This factorization is not unique, but one factorization that is particularly useful is

$$[I, \hat{\gamma}_{23}, \hat{\gamma}_{31}, \hat{\gamma}_{12}] \times [I, \hat{\gamma}_0, J, J\hat{\gamma}_0] \quad \text{where } J = \hat{\gamma}_{0123}.$$

Note several properties. First: every element in the first set commutes with every element in the second set. Second: linear combinations of elements in the first set form the algebra of quaternions. (This is obvious since we can identify $\hat{\gamma}_{23}$ with $\boldsymbol{i}$, $\hat{\gamma}_{31}$ with $\boldsymbol{j}$, and $\hat{\gamma}_{12}$ with $\boldsymbol{k}$.) Third: linear combinations of elements of the second set form the algebra of real 2×2 matrices. This is true since we can represent

$$I \text{ by } \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}, \qquad \hat{\gamma}_0 \text{ by } \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix},$$

$$J \quad \text{by} \quad \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \qquad \text{and} \qquad J\hat{\gamma}_0 \quad \text{by} \quad \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

In this representation

$$aI + b\hat{\gamma}_0 + cJ + dJ\hat{\gamma}_0 = \begin{bmatrix} a+d & b+c \\ b-c & a-d \end{bmatrix}.$$

Thus the algebra generated by I, $\hat{\gamma}_0$, J, and $J\hat{\gamma}_0$ over the field of real numbers is isomorphic to the algebra of real 2×2 matrices.

Any Clifford number associated with Minkowski 4-space can be written as a linear combination of elements formed by taking products of elements in the quaternion algebra with elements in the 2×2 real total matric algebra. For example an arbitrary boost operator $\mathscr{B}$ can be written in the form

$$\mathscr{B} = I\cosh\frac{\phi}{2} + J\hat{k}\sinh\frac{\phi}{2} = \exp(\tfrac{1}{2}\phi J\hat{k}) \tag{A1.1}$$

where

$$\hat{k} = k^1\hat{\gamma}_{23} + k^2\hat{\gamma}_{31} + k^3\hat{\gamma}_{12}, \tag{A1.2}$$

$$J\hat{k} = k^1\hat{\gamma}_{10} + k^2\hat{\gamma}_{20} + k^3\hat{\gamma}_{30}, \tag{A1.3}$$

and (k^1, k^2, k^3) are the direction cosines of the direction of the boost.

Similarly, a rotation operator $\mathscr{R}$ may be written in the form

$$\mathscr{R} = I\cos\frac{\theta}{2} + \hat{n}\sin\frac{\theta}{2} = \exp(\tfrac{1}{2}\theta\hat{n}) \tag{A1.4}$$

where $\hat{n} = n^1\hat{\gamma}_{23} + n^1\hat{\gamma}_{31} + n^3\hat{\gamma}_{12}$, and (n^1, n^2, n^3) are the direction cosines of the axis of rotation.

The bivectors $\hat{k}$, $J\hat{k}$, and $\hat{n}$ are said to be *simple.* In general any Clifford number is said to be simple if it can be written as a product of vectors. It is obvious that $J\hat{k}$ is simple since $J\hat{k} = (k^1\hat{\gamma}_1 + k^2\hat{\gamma}_2 + k^3\hat{\gamma}_3)\hat{\gamma}_0$. It is less obvious that $\hat{k}$ or $\hat{n}$ is simple. However, if one chooses two unit vectors (p^1, p^2, p^3) and (q^1, q^2, q^3) which are orthogonal to (k^1, k^2, k^3) and orthogonal to one another then

$$(k^1, k^2, k^3) = \pm(p^2q^3 - p^3q^2, p^3q^1 - p^1q^3, p^1q^2 - p^2q^1)$$

and

$$\hat{k} = \pm(p^1\hat{\gamma}_1 + p^2\hat{\gamma}_2 + p^3\hat{\gamma}_3)(q^1\hat{\gamma}_1 + q^2\hat{\gamma}_2 + q^3\hat{\gamma}_3).$$

In the following exposition, I will refer to normalized bivectors similar to $\hat{n}$ and $\hat{k}$ as *simple unit space–space bivectors* and normalized bivectors such

as $J\hat{k}$ as *simple unit space–time bivectors.* Note that

$$(\hat{n})^2 = -[(n^1)^2 + (n^2)^2 + (n^3)^2] = -1.$$

The advantage of using Wedderburn's decomposition for computational purposes is that one only needs to remember the products of the members of the quaternion algebra and the matric algebra separately. For the quaternion algebra, one only needs to remember the following equation which can be verified by the reader; namely

$$\hat{n}\hat{k} = -I(\hat{n}\cdot\hat{k}) + \hat{n} \times \hat{k} \tag{A1.5}$$

where

$$(\hat{n}\cdot\hat{k}) = n^1k^1 + n^2k^2 + n^3k^3$$

and

$$\hat{n} \times \hat{k} = (n^2k^3 - n^3k^2)\hat{\gamma}_{23} + (n^3k^1 - n^1k^3)\hat{\gamma}_{31} + (n^1k^2 - n^2k^1)\hat{\gamma}_{12}.$$

(The reader should be careful to distinguish Eq. (A1.5), which is an equation for the product of 2-vectors in Minkowski 4-space, from Eq. (1.6), which is an equation for the product of 1-vectors in Euclidean 3-space.)

To carry out computations in the matric component of the algebra, one only needs to remember $(\hat{\gamma}_0)^2 = I$, $J^2 = -I$, and $\hat{\gamma}_0 J = -J\hat{\gamma}_0$. Thus for example, $(J\hat{\gamma}_0)(J\hat{\gamma}_0) = -(\hat{\gamma}_0 J)(J\hat{\gamma}_0) = -\hat{\gamma}_0(J^2)\hat{\gamma}_0 = (\hat{\gamma}_0)^2 = I$. With a little practice this can be done quite quickly in one's head.

An example of a slightly more complex calculation follows: suppose $\mathcal{M}$ is an arbitrary Clifford number consisting of a linear combination of vectors of even order. Suppose we wish to compute $\mathcal{M}\mathcal{M}^\dagger$, where $\mathcal{M}^\dagger$ is the Clifford reverse of $\mathcal{M}$. We may represent $\mathcal{M}$ in the form

$$\mathcal{M} = aI + b\hat{n} + cJ\hat{k} + dJ \tag{A1.6}$$

where $\hat{n}$ and $\hat{k}$ are normalized so that $\hat{n}\hat{n} = \hat{k}\hat{k} = -(\hat{n}\cdot\hat{n}) = -1$. Then

$$\begin{aligned}\mathcal{M}^\dagger &= aI + b\hat{n}^\dagger + ck^\dagger J^\dagger + dJ^\dagger \\ &= aI + b(-\hat{n}) + c(-\hat{k})(J) + dJ \\ &= aI - b\hat{n} - cJ\hat{k} + dJ.\end{aligned} \tag{A1.7}$$

Carrying out the multiplication, we find that many terms cancel out and we get

$$\mathcal{M}\mathcal{M}^\dagger = I(a^2 + b^2 - c^2 - d^2) + J(2ad + 2bc(\hat{n}\cdot\hat{k})). \tag{A1.8}$$

We are now in a good position to prove our first theorem.

Theorem A1.1. Suppose $\mathscr{M}$ is a real Clifford number associated with Minkowski 4-space and is composed of a linear combination of even order vectors. Also suppose $\mathscr{M}\mathscr{M}^\dagger \neq 0$. Then $\mathscr{M} = N\mathscr{L}\mathscr{J}$ where N is a positive normalizing constant, $\mathscr{L}$ is a linear combination of even forms with the property that $\mathscr{L}\mathscr{L}^\dagger = I$, and $\mathscr{J} = \exp(\alpha J/2) = I\cos\alpha/2 + J\sin\alpha/2$. (In his book *Space–Time Algebra*, David Hestenes refers to the operator $\mathscr{J}$ as a *duality rotation* (1966, p. 17).)

Proof. Referring to Eqs. (A1.6)–(A1.8), we can let

$$N^2 = [(a^2 + b^2 - c^2 - d^2)^2 + 4(ad + bc(\hat{k}\cdot\hat{n}))^2]^{\frac{1}{2}},$$

$$\cos\alpha = (1/N^2)(a^2 + b^2 - c^2 - d^2),$$

and

$$\sin\alpha = (1/N^2)2(ad + bc(\hat{k}\cdot\hat{n})).$$

Then

$$\mathscr{M}\mathscr{M}^\dagger = N^2(I\cos\alpha + J\sin\alpha) = N^2\exp(\alpha J).$$

Now let $N^2\mathscr{J}^2 = \mathscr{M}\mathscr{M}^\dagger = N^2\exp(\alpha J)$ and $\mathscr{J} = \pm\exp(\alpha J/2)$. Clearly the inverse operator $\mathscr{J}^{-1}$ exists. In fact $\mathscr{J}^{-1} = \pm\exp(-\alpha J/2)$. Now define $\mathscr{L} = (1/N)\mathscr{M}\mathscr{J}^{-1}$. Then $\mathscr{L}^\dagger = (1/N)(\mathscr{J}^{-1})^\dagger\mathscr{M}^\dagger$. But $(\mathscr{J}^{-1})^\dagger = \mathscr{J}^{-1}$. Also J and therefore $\mathscr{J}^{-1}$ commutes with all even order vectors. Thus

$$\mathscr{L}\mathscr{L}^\dagger = (1/N^2)\mathscr{M}\mathscr{M}^\dagger\mathscr{J}^{-2} = (1/N^2)(N^2\mathscr{J}^2)\mathscr{J}^{-2} = I.$$

I leave it to the reader to show that aside from the sign ambiguity $\mathscr{J}$ and $\mathscr{L}$ are uniquely determined by $\mathscr{M}$. ■

Having factored out $\mathscr{J}$ from $\mathscr{M}$, we can now turn to the problem of factoring the remaining factor $\mathscr{L}$.

Theorem A1.2. Suppose $\mathscr{L}$ is a linear combination of even order vectors such that $\mathscr{L}\mathscr{L}^\dagger = I$. Then $\mathscr{L} = \mathscr{B}\mathscr{R}$ where $\mathscr{B}$ is a boost operator and $\mathscr{R}$ is a rotation operator.

Proof. To get a grip on what we are doing, it is useful to multiply a boost and a rotation to see what the product looks like:

$$\begin{aligned}\left(I\cosh\frac{\phi}{2} + J\hat{m}\sinh\frac{\phi}{2}\right)&\left(I\cos\frac{\theta}{2} + \hat{n}\sin\frac{\theta}{2}\right)\\ &= I\cosh\frac{\phi}{2}\cos\frac{\theta}{2} + \cosh\frac{\phi}{2}\sin\frac{\theta}{2}\hat{n} + J\hat{m}\sinh\frac{\phi}{2}\cos\frac{\theta}{2}\\ &\quad + J\hat{m}\times\hat{n}\sinh\frac{\phi}{2}\sin\frac{\theta}{2} - J(\hat{m}\cdot\hat{n})\sinh\frac{\theta}{2}\sin\frac{\theta}{2}. \qquad \text{(A1.9)}\end{aligned}$$

By examining the first two terms on the right-hand side of Eq. (A1.9), one can determine the rotation component of any restricted Lorentz operator by inspection. Thus if

$$\mathscr{L} = aI + b\hat{n} + cJ\hat{k} + dJ$$

then

$$\mathscr{R} = I\cos\frac{\theta}{2} + \hat{n}\sin\frac{\theta}{2}$$

where

$$\cos\frac{\theta}{2} = \frac{a}{\sqrt{a^2+b^2}} \qquad \text{and} \qquad \sin\frac{\theta}{2} = \frac{b}{\sqrt{a^2+b^2}}.$$

To show that the factoring can be done, we only need to show that $\mathscr{L}\mathscr{R}^{-1}$ is a boost:

$$\begin{aligned}
\mathscr{L}\mathscr{R}^{-1} &= \frac{1}{\sqrt{a^2+b^2}}(aI + b\hat{n} + cJ\hat{k} + dJ)(aI - b\hat{n}) \\
&= \frac{1}{\sqrt{a^2+b^2}}[(a^2+b^2)I + J(ac\hat{k} - bc\hat{k}\hat{n} + ad - bd\hat{n})] \\
&= \frac{1}{\sqrt{a^2+b^2}}[(a^2+b^2)I + J(ac\hat{k} - bc\hat{k}\times\hat{n} - bd\hat{n}) \\
&\quad + J(ad + bc(\hat{k}\cdot\hat{n}))].
\end{aligned} \tag{A1.10}$$

Since $\mathscr{L}\mathscr{L}^{\dagger} = I$, it follows from Eq. (A1.8) that $ad + bc(\hat{k}\cdot\hat{n}) = 0$. What remains on the right-hand side of Eq. (A1.10) looks hopefully like a boost. To push this further, let us examine the magnitude of the bivector. Suppose $\boldsymbol{v} = ac\hat{k} - bc(\hat{k}\times\hat{n}) - bd\hat{n}$. Then

$$\begin{aligned}
(\boldsymbol{v}\cdot\boldsymbol{v}) &= a^2c^2(\hat{k}\cdot\hat{k}) + b^2c^2|\hat{k}\times\hat{n}|^2 + b^2d^2(\hat{n}\cdot\hat{n}) - 2acbd(\hat{n}\cdot\hat{k}) \\
&= a^2c^2 + b^2c^2(1 - (\hat{k}\cdot\hat{n})^2) + b^2d^2 - 2ad(bc(\hat{n}\cdot\hat{k})).
\end{aligned}$$

Using Eq. (A1.8) again, we have $bc(\hat{k}\cdot\hat{n}) = -ad$ and

$$\begin{aligned}
(\boldsymbol{v}\cdot\boldsymbol{v}) &= a^2c^2 + b^2c^2 - a^2d^2 + b^2d^2 + 2a^2d^2 \\
&= a^2c^2 + b^2c^2 + a^2d^2 + b^2d^2 \\
&= (a^2+b^2)(c^2+d^2).
\end{aligned}$$

Then

$$(ac\hat{k} - bc(\hat{k} \times \hat{n}) - bd\hat{n}) = \sqrt{(a^2 + b^2)(c^2 + d^2)}\,\hat{m}$$

where $\hat{m}$ is a simple unit space–space bivector. Using this result Eq. (A1.10) becomes

$$\mathscr{L}\mathscr{R}^{-1} = \sqrt{a^2 + b^2}\,I + \sqrt{c^2 + d^2}\,J\hat{m}.$$

To identify this as a boost operator, we need to make the identification that

$$\sqrt{a^2 + b^2} = \cosh\left(\frac{\phi}{2}\right) \quad \text{and} \quad \sqrt{c^2 + d^2} = \sinh\left(\frac{\phi}{2}\right)$$

for some appropriate value for ϕ. Since $\cosh^2(\phi/2) - \sinh^2(\phi/2) = 1$, our desired identification is possible only if $(a^2 + b^2) - (c^2 + d^2) = 1$. However, looking at Eq. (A1.8) again we see that $\mathscr{L}\mathscr{L}^\dagger = I$ implies that $(a^2 + b^2) - (c^2 + d^2) = 1$. Thus we do indeed have our sought-after boost operator. ■

It should be noted that a restricted Lorentz operator can also be factored with the rotation operator on the left side. In that case the rotation operator will be the same but the direction of the boost operator will be different unless the direction of the boost is the same as the direction of the axis of rotation. To see this we note that

$$\mathscr{B}(\hat{m}, \phi)\mathscr{R}(\hat{n}, \theta) = \mathscr{R}(\mathscr{R}^{-1}\mathscr{B}\mathscr{R}) = \mathscr{R}\mathscr{B}(\hat{m}', \phi)$$

where

$$\hat{m}' = \mathscr{R}^{-1}(\hat{n}, \theta)\hat{m}\mathscr{R}(\hat{n}, \theta).$$

Combining Theorems A1.1 and A1.2, we see that if $\mathscr{M}\mathscr{M}^\dagger \neq 0$, then $\mathscr{M} = N\mathscr{B}\mathscr{R}\mathscr{J}$, where N is a positive normalizing constant, $\mathscr{B}$ is a boost operator, $\mathscr{R}$ is a rotation operator, and $\mathscr{J}$ is a duality rotation. To complete the picture, let us consider the case where $\mathscr{M}\mathscr{M}^\dagger = 0$.

Theorem A1.3. Suppose $\mathscr{M}$ is a linear combination of even order p-vectors associated with Minkowski 4-space and $\mathscr{M}\mathscr{M}^\dagger = 0$. Then $\mathscr{M} = N\mathscr{R}\mathscr{P}$, where N is a positive normalizing constant, $\mathscr{R}$ is a rotation operator, and $\mathscr{P}$ is a projection operator of the form $\frac{1}{2}(I + J\hat{m})$ where $\hat{m}$ is a simple unit space–space bivector. Furthermore N, $\mathscr{R}$ and $\mathscr{J}$ are unique. (An operator $\mathscr{P}$ is said to be a *projection operator* if $\mathscr{P}^2 = \mathscr{P}$.)

Proof. By examining a product of the form $N\mathscr{R}\mathscr{P}$, one can pick out the rotation operator by inspection and the rest rapidly ensues. Suppose

$\mathscr{R} = \exp(\frac{1}{2}\theta\hat{n})$ and $\mathscr{P} = \frac{1}{2}(I + J\hat{m})$. Then

$$N\mathscr{R}\mathscr{P} = \frac{N}{2}\left[I\cos\frac{\theta}{2} + \hat{n}\sin\frac{\theta}{2} + J\left(\hat{m}\cos\frac{\theta}{2} + (\hat{n}\times\hat{m})\sin\frac{\theta}{2}\right) - J(\hat{n}\cdot\hat{m})\sin\frac{\theta}{2}\right].$$

Thus if $\mathscr{M} = aI + b\hat{n} + cJ\hat{k} + dJ$, inspection of the first two terms tells us that

$$\mathscr{R} = \frac{a}{\sqrt{a^2+b^2}}I + \frac{b}{\sqrt{a^2+b^2}}\hat{n}.$$

To obtain the projection factor, we compute

$$\begin{aligned}\mathscr{R}^{-1}\mathscr{M} &= \frac{1}{\sqrt{a^2+b^2}}(aI - b\hat{n})(aI + b\hat{n} + cJ\hat{k} + dJ)\\ &= \frac{1}{\sqrt{a^2+b^2}}[(a^2+b^2)I + J(ac\hat{k} - bc(\hat{n}\times\hat{k}) - bd\hat{n})\\ &\quad + J(ad + bc(\hat{n}\cdot\hat{k}))].\end{aligned}$$

Following steps virtually identical to those in the proof of Theorem A1.2, we get

$$\mathscr{R}^{-1}\mathscr{M} = \frac{1}{\sqrt{a^2+b^2}}[(a^2+b^2)I + (c^2+d^2)J\hat{m}]$$

where

$$\hat{m} = \frac{ac\hat{k} - bc(\hat{n}\times\hat{k}) - bd\hat{n}}{\sqrt{(a^2+b^2)(c^2+d^2)}}.$$

However, using Eq. (A1.8), $\mathscr{M}\mathscr{M}^\dagger = 0$ implies that $(a^2+b^2) = (c^2+d^2)$. Thus $\mathscr{R}^{-1}\mathscr{M} = N\frac{1}{2}(I + J\hat{m})$ where $N = 2\sqrt{a^2+b^2}$. ■

A.2 The Exponential Representation of Restricted Lorentz Operators

In the first section of this appendix, it was shown that a restricted Lorentz operator can be decomposed into a product of a boost and a rotation operator. In this section, it will be shown that a restricted Lorentz operator can be represented as a single exponential operator with a bivector argument. In proving this assertion, it will be shown how to construct the representation. From this representation, it will be shown how to read off the eigenvectors. Our first theorem to be proven is:

Theorem A2.1. Suppose $\mathscr{L}$ is a restricted Lorentz transformation. In

particular suppose $\mathscr{L} = aI + b\hat{n} + cJ\hat{k} + dJ$. Then one of the two following cases hold.

(1) $(b\hat{n} + cJ\hat{k})^2 = 0$ and

$$\begin{aligned} \mathscr{L} &= \exp(b\hat{n}(I - J\hat{m})) \qquad \text{or} \qquad \mathscr{L} = -\exp(-b\hat{n}(I - J\hat{m})) \\ &= \pm I + b\hat{n}(I - J\hat{m}), \end{aligned} \tag{A2.1}$$

where $\hat{m} = \hat{n}\hat{k} = \hat{n} \times \hat{k}$ and the three entities $\hat{k}$, $\hat{m}$, and $\hat{n}$ form a right-hand system of orthonormal space–space bivectors. If we identify $\hat{k} = \hat{\gamma}_{23}$, $\hat{m} = \hat{\gamma}_{31}$, and $\hat{n} = \hat{\gamma}_{12}$, then

$$\mathscr{L} = \exp(b\hat{\gamma}_1(\hat{\gamma}_0 + \hat{\gamma}_2)) \qquad \text{or} \qquad \mathscr{L} = -\exp(-\mathrm{b}(\hat{\gamma}_1(\hat{\gamma}_0 + \hat{\gamma}_2)). \tag{A2.2}$$

(2) $(b\hat{n} + cJ\hat{k})^2 \neq 0$ and

$$\mathscr{L} = \exp(\tfrac{1}{2}(\phi - \theta J)\hat{z}), \tag{A2.3}$$

where $\hat{z}$ is a "complex multiple" of $(b\hat{n} + cJ\hat{k})$ such that $(\hat{z})^2 = I$. To say $\hat{z}$ is a "complex multiple" of $(b\hat{n} + cJ\hat{k})$, I mean that $\hat{z} = (1/r)\exp(-\alpha J/2)(b\hat{n} + cJ\hat{k})$, where $\exp(-\alpha J/2)$ is considered "complex" because in this context J is algebraically equivalent to $\sqrt{-1}$. For this second case it is possible to express $\mathscr{L}$ in the form:

$$\mathscr{L} = \exp(\tfrac{1}{2}(\phi\hat{\gamma}_{10} + \theta\hat{\gamma}_{23}). \tag{A2.4}$$

However, in general, this form is valid only in a boosted frame and not in the frame of the observer.

Proof for case 1.

$$\mathscr{L} = (aI + dJ) + (b\hat{n} + cJ\hat{k}), \qquad \mathscr{L}^\dagger = (aI + dJ) - (b\hat{n} + cJ\hat{k}),$$

and

$$\mathscr{L}\mathscr{L}^\dagger = (aI + dJ)^2 = (a^2 - d^2)I + adJ = I.$$

Since $a^2 = d^2 + 1$, we know that $a \neq 0$ and since $2ad = 0$, we know that $d = 0$. It then follows that $a = \pm 1$. Furthermore $(b\hat{n} + cJ\hat{k})^2 = -b^2 + c^2 - 2bcJ(\hat{n}\cdot\hat{k}) = 0$. Thus either both b and c are zero or neither is zero and $(\hat{n}\cdot\hat{k}) = 0$. In the first case $\mathscr{L} = \pm I$. In the second case, $c = \pm b$. Absorbing the sign ambiguity of c into the definition of $\hat{k}$, we have

$$(b\hat{n} + cJ\hat{k}) = (b\hat{n} + bJ\hat{k}) = b\hat{n}(I - J\hat{n}\hat{k}) = b\hat{n}(I - J\hat{m})$$

and

$$\mathscr{L} = \pm I + b\hat{n}(I - J\hat{m}) = \pm\exp(\pm b\hat{n}(I - J\hat{m})).$$

The reader should check that $\hat{k}$, $\hat{m}$, and $\hat{n}$ form a right-handed system of orthonormal bivectors, that is $\hat{n} \times \hat{m} = \hat{m}\hat{n} = -\hat{n}\hat{m} = \hat{k}$, $\hat{n} \times \hat{k} = \hat{n}\hat{k} = -\hat{k}\hat{n} = \hat{m}$, and $\hat{k} \times \hat{m} = \hat{k}\hat{m} = -\hat{m}\hat{k} = \hat{n}$.

Proof for case 2. For case 2, it is helpful to observe that in this context J is algebraically equivalent to $\sqrt{-1}$. Thus we can use our knowledge of complex numbers to carry out our computations. We first observe that $(b\hat{n} + cJ\hat{k})^2 = (-b^2 + c^2)I - 2bc(\hat{k}\cdot\hat{n})J$ which is much like an ordinary complex number. Thus if we let

$$(-b^2 + c^2)I - 2bc(\hat{k}\cdot\hat{n})J = r^2(I\cos\alpha + J\sin\alpha) = r^2\exp(\alpha J), \quad \text{(A2.5)}$$

and define

$$\hat{z} = \frac{1}{r}(b\hat{n} + cJ\hat{k})\exp\left(-\frac{\alpha J}{2}\right), \quad \text{(A2.6)}$$

then it is not difficult to show that $(\hat{z})^2 = I$.

With these definitions, we have

$$\mathscr{L} = aI + dJ + r\exp\left(\frac{\alpha J}{2}\right)\hat{z} \quad \text{(A2.7)}$$

or

$$\mathscr{L} = A + B\hat{z}. \quad \text{(A2.8)}$$

We observe that since $\mathscr{L}\mathscr{L}^\dagger = (A + B\hat{z})(A - B\hat{z})$,

$$A^2 - B^2 = I. \quad \text{(A2.9)}$$

From Eq. (A2.9), we can identify A and B with the hyperbolic cosine and sine functions with "complex" arguments, that is

$$A = \cosh(\tfrac{1}{2}(\phi - \theta J)) \quad \text{(A2.10)}$$

and

$$B = \sinh(\tfrac{1}{2}(\phi - \theta J)). \quad \text{(A2.11)}$$

With these definitions, we have

$$\begin{aligned} \mathscr{L} &= \cosh(\tfrac{1}{2}(\phi - \theta J)) + \hat{z}\sinh(\tfrac{1}{2}(\phi - \theta J)) \\ &= \exp(\tfrac{1}{2}(\phi - \theta J)\hat{z}). \end{aligned} \quad \text{(A2.12)}$$

To determine ϕ and θ, we can expand the right-hand sides of Eqs. (A2.10) and (A2.11) and compare the results with the right-hand side of Eq. (A2.7). Thus we have

$$\begin{aligned} A &= I\cosh\left(\frac{\phi}{2}\right)\cosh\left(\frac{\theta J}{2}\right) - \sinh\left(\frac{\phi}{2}\right)\sinh\left(\frac{\theta J}{2}\right) \\ &= I\cosh\left(\frac{\phi}{2}\right)\cos\left(\frac{\theta}{2}\right) - J\sinh\left(\frac{\phi}{2}\right)\sin\left(\frac{\theta}{2}\right) \\ &= aI + dJ, \end{aligned} \tag{A2.13}$$

and

$$\begin{aligned} B &= I\sinh\left(\frac{\phi}{2}\right)\cosh\left(\frac{\theta J}{2}\right) - \cosh\left(\frac{\phi}{2}\right)\sinh\left(\frac{\theta J}{2}\right) \\ &= I\sinh\left(\frac{\phi}{2}\right)\cos\left(\frac{\theta}{2}\right) - J\cosh\left(\frac{\phi}{2}\right)\sin\left(\frac{\theta}{2}\right) \\ &= Ir\cos\left(\frac{\alpha}{2}\right) + Jr\sin\left(\frac{\alpha}{2}\right). \end{aligned} \tag{A2.14}$$

Thus we have

$$\cosh\left(\frac{\phi}{2}\right)\cos\left(\frac{\theta}{2}\right) = a, \qquad \sinh\left(\frac{\phi}{2}\right)\sin\left(\frac{\theta}{2}\right) = -d, \quad \text{(A2.15), (A2.16)}$$

$$\sinh\left(\frac{\phi}{2}\right)\cos\left(\frac{\theta}{2}\right) = r\cos\left(\frac{\alpha}{2}\right), \tag{A2.17}$$

and

$$\cosh\left(\frac{\phi}{2}\right)\sin\left(\frac{\theta}{2}\right) = -r\sin\left(\frac{\alpha}{2}\right). \tag{A2.18}$$

Taking ratios of these equations, we have

$$\tanh\left(\frac{\phi}{2}\right) = \frac{r\cos(\alpha/2)}{a} = \frac{d}{r\sin(\alpha/2)}, \tag{A2.19}$$

and

$$\tan\left(\frac{\theta}{2}\right) = -\frac{r\sin(\alpha/2)}{a} = \frac{-d}{r\cos(\alpha/2)}. \tag{A2.20}$$

The argument ϕ is uniquely determined by Eq. (A2.19). The argument θ is uniquely determined by Eq. (A2.20) if we require that $-\pi/2 < \theta/2 \leqslant \pi/2$ when a is nonnegative and $\pi/2 < \theta/2 < 3\pi/2$ when a is negative.

We now turn to the problem of showing that we can write $\mathscr{L}$ as

$\exp(\frac{1}{2}(\phi\hat{\gamma}_{10} + \theta\hat{\gamma}_{23}))$. To do this we have to examine the form of $\hat{z}$ more closely. From Eq. (A2.6), we have

$$\hat{z} = \frac{1}{r}\left[b\cos\left(\frac{\alpha}{2}\right)\hat{n} + c\sin\left(\frac{\alpha}{2}\right)\hat{k} + J\left(-b\sin\left(\frac{\alpha}{2}\right)\hat{n} + c\cos\left(\frac{\alpha}{2}\right)\hat{k}\right)\right] \quad \text{(A2.21)}$$

or

$$\hat{z} = \bar{x} + J\bar{y}$$

where

$$\bar{x} = \frac{1}{r}\left(b\cos\left(\frac{\alpha}{2}\right)\hat{n} + c\sin\left(\frac{\alpha}{2}\right)\hat{k}\right) \quad \text{(A2.22)}$$

and

$$\bar{y} = \frac{1}{r}\left(-b\sin\left(\frac{\alpha}{2}\right)\hat{n} + c\cos\left(\frac{\alpha}{2}\right)\hat{k}\right). \quad \text{(A2.23)}$$

From these equations,

$$\begin{aligned} r^2(\bar{x}\cdot\bar{y}) &= (c^2 - b^2)\sin\left(\frac{\alpha}{2}\right)\cos\left(\frac{\alpha}{2}\right) + bc(\hat{n}\cdot\hat{k})\left(\cos^2\left(\frac{\alpha}{2}\right) - \sin^2\left(\frac{\alpha}{2}\right)\right) \\ &= \tfrac{1}{2}(c^2 - b^2)\sin\alpha + bc(\hat{k}\cdot\hat{n})\cos\alpha. \end{aligned}$$

From Eq. (A2.5),

$$(c^2 - b^2) = r^2\cos\alpha \qquad \text{and} \qquad bc(\hat{k}\cdot\hat{n}) = -\tfrac{1}{2}r^2\sin\alpha.$$

Thus

$$(\bar{x}\cdot\bar{y}) = 0. \quad \text{(A2.24)}$$

In addition

$$\begin{aligned} (\bar{x}\cdot\bar{x}) &= \left(\frac{1}{r}\right)^2\left[b^2\cos^2\left(\frac{\alpha}{2}\right) + c^2\sin^2\left(\frac{\alpha}{2}\right) + 2bc(\hat{k}\cdot\hat{n})\sin\left(\frac{\alpha}{2}\right)\cos\left(\frac{\alpha}{2}\right)\right] \\ &= \left(\frac{1}{r}\right)^2\left[\frac{b^2(1+\cos\alpha)}{2} + \frac{c^2(1-\cos\alpha)}{2} + bc(\hat{k}\cdot\hat{n})\sin\alpha\right] \\ &= \left(\frac{1}{r}\right)^2\left[\frac{b^2+c^2}{2} + \frac{b^2-c^2}{2}\cos\alpha + bc(\hat{k}\cdot\hat{n})\sin\alpha\right] \\ &= \left(\frac{1}{r}\right)^2\left[\frac{b^2+c^2}{2} - \frac{r^2}{2}\cos^2\alpha - \frac{r^2}{2}\sin^2\alpha\right] \end{aligned}$$

or

$$(\bar{x}\cdot\bar{x}) = \frac{b^2 + c^2 - r^2}{2r^2}. \quad \text{(A2.25)}$$

A similar calculation shows that

$$(\bar{y}\cdot\bar{y}) = \frac{b^2 + c^2 + r^2}{2r^2}. \tag{A2.26}$$

Since the right-hand sides of (A2.25) and (A2.26) are both positive and their difference is 1, we can define

$$(\bar{x}\cdot\bar{x}) = \sinh^2\beta \qquad \text{and} \qquad (\bar{y}\cdot\bar{y}) = \cosh^2\beta.$$

Furthermore since $\bar{x}$ and $\bar{y}$ are orthogonal space–space bivectors, we can choose an orientation for our frame such that

$$\bar{x} = \hat{\gamma}_{12}\sinh\beta \qquad \text{and} \qquad \bar{y} = \hat{\gamma}_{23}\cosh\beta.$$

This means that

$$\begin{aligned}\hat{z} = \bar{x} + J\bar{y} &= \hat{\gamma}_{12}\sinh\beta + \hat{\gamma}_{10}\cosh\beta\\ &= \hat{\gamma}_1(\hat{\gamma}_0\cosh\beta + \hat{\gamma}_2\sinh\beta).\end{aligned} \tag{A2.27}$$

In a boosted frame, we have

$$\begin{aligned}\hat{\gamma}'_0 &= \hat{\gamma}_0\cosh\beta + \hat{\gamma}_2\sinh\beta,\\ \hat{\gamma}'_1 &= \hat{\gamma}_1\\ \hat{\gamma}'_2 &= \hat{\gamma}_0\sinh\beta + \hat{\gamma}_2\cosh\beta,\\ \hat{\gamma}'_3 &= \hat{\gamma}_3.\end{aligned}$$

In this frame, we have

$$\hat{z} = \hat{\gamma}'_{10} \qquad \text{and} \qquad J\hat{z} = -\hat{\gamma}'_{23}.$$

Dropping the primes, Eq. (A2.12) becomes

$$\mathscr{L} = \exp(\tfrac{1}{2}(\phi\hat{\gamma}_{10} + \theta\hat{\gamma}_{23})). \qquad \blacksquare \tag{A2.28}$$

Using the representation $\mathscr{L} = \exp(\frac{1}{2}(\phi - \theta J)\hat{z})$, one can obtain explicit formulas for the eigenvalues and eigenvectors. In particular we have:

Theorem A2.2. If $\mathscr{L} = \exp(\frac{1}{2}(\phi - \theta J)\hat{z})$, then

$$\mathscr{L}[\hat{\gamma}_0(\bar{y}\cdot\bar{y}) + J\hat{\gamma}_0\bar{x}\bar{y} \pm J\hat{\gamma}_0\bar{y}]\mathscr{L}^\dagger = \mathrm{e}^{\pm\phi}[\hat{\gamma}_0(\bar{y}\cdot\bar{y}) + J\hat{\gamma}_0\bar{x}\bar{y} \pm J\hat{\gamma}_0\bar{y}] \tag{A2.29}$$

and

$$\mathscr{L}[\hat{\gamma}_0(\bar{x}\cdot\bar{x}) + J\hat{\gamma}_0\bar{x}\bar{y} \pm \mathrm{i}J\hat{\gamma}_0\bar{x}]\mathscr{L}^\dagger = \mathrm{e}^{\mp \mathrm{i}\theta}[\hat{\gamma}_0(\bar{x}\cdot\bar{x}) + J\hat{\gamma}_0\bar{x}\bar{y} \pm \mathrm{i}J\hat{\gamma}_0\bar{x}]. \tag{A2.30}$$

Proof. To check these equations, one needs to compute $\mathscr{L}\hat{\gamma}_0\mathscr{L}^\dagger$, $\mathscr{L}J\hat{\gamma}_0\bar{x}\mathscr{L}^\dagger$, $\mathscr{L}J\hat{\gamma}_0\bar{y}\mathscr{L}^\dagger$, and $\mathscr{L}J\hat{\gamma}_0\bar{x}\bar{y}\mathscr{L}^\dagger$. To compute $\mathscr{L}\hat{\gamma}_0\mathscr{L}^\dagger$, we note that $\hat{\gamma}_0$ commutes with both $\bar{x}$ and $\bar{y}$ but anticommutes with J. Thus

$$\begin{aligned}\mathscr{L}\hat{\gamma}_0\mathscr{L}^\dagger &= (\exp(\tfrac{1}{2}(\phi - \theta J)z)\hat{\gamma}_0 \exp(-\tfrac{1}{2}(\phi - \theta J)\hat{z})\\ &= \hat{\gamma}_0 \exp(\tfrac{1}{2}(\phi + \theta J)\hat{z}^*) \exp(-\tfrac{1}{2}(\phi - \theta J)\hat{z}).\end{aligned}$$

Here it is understood that $\hat{z}^* = \bar{x} - J\bar{y}$. We now have

$$\begin{aligned}\mathscr{L}\hat{\gamma}_0\mathscr{L}^\dagger = \hat{\gamma}_0[&\cosh\tfrac{1}{2}(\phi + \theta J) + \hat{z}^* \sinh\tfrac{1}{2}(\phi + \theta J)]\\ &\times [\cosh\tfrac{1}{2}(\phi - \theta J) - \hat{z}\sinh\tfrac{1}{2}(\phi - \theta J)].\end{aligned}$$

We note that

$$\begin{aligned}\hat{z}^*\hat{z} &= (\bar{x} - J\bar{y})(\bar{x} + J\bar{y}) = \bar{x}\bar{x} + \bar{y}\bar{y} + J(\bar{x}\bar{y} - \bar{y}\bar{x})\\ &= -(\bar{x}\cdot\bar{x}) - (\bar{y}\cdot\bar{y}) + 2J\bar{x}\bar{y}.\end{aligned}$$

Also

$$\begin{aligned}\cosh A \cosh B &= \tfrac{1}{2}[\cosh(A + B) + \cosh(A - B)],\\ \sinh A \sinh B &= \tfrac{1}{2}[\cosh(A + B) - \cosh(A - B)],\end{aligned}$$

and

$$\sinh A \cosh B = \tfrac{1}{2}[\sinh(A + B) + \sinh(A - B)].$$

This gives us

$$\begin{aligned}\mathscr{L}\hat{\gamma}_0\mathscr{L}^\dagger = \hat{\gamma}_0[&\tfrac{1}{2}(\cosh\phi + \cosh(\theta J))\\ &+ ((\bar{x}\cdot\bar{x}) + (\bar{y}\cdot\bar{y}))\tfrac{1}{2}(\cosh\phi - \cosh(\theta J))\\ &- 2J\bar{x}\bar{y}\tfrac{1}{2}(\cosh\phi - \cosh(\theta J))\\ &+ \hat{z}^*\tfrac{1}{2}(\sinh\phi + \sinh(\theta J)) - \hat{z}\tfrac{1}{2}(\sinh\phi + \sinh(-\theta J))].\end{aligned}$$

We observe that $\cosh(\theta J) = I\cos\theta$, $\sinh(\theta J) = J\sin\theta$, and $\sinh(-\theta J) = -\sinh(\theta J) = -J\sin\theta$. Also $(\bar{y}\cdot\bar{y}) - (\bar{x}\cdot\bar{x}) = 1$. From these equations, we have

$$\begin{aligned}\mathscr{L}\hat{\gamma}_0\mathscr{L}^\dagger = \hat{\gamma}_0[&(\bar{y}\cdot\bar{y})\cosh\phi - (\bar{x}\cdot\bar{x})\cos\theta\\ &- J\bar{x}\bar{y}(\cosh\phi - \cos\theta) + J\bar{x}\sin\theta - J\bar{y}\sinh\phi].\end{aligned}$$

Multiplying out the $\hat{\gamma}_0$ and reordering the terms, gives us

$$\begin{aligned}\mathscr{L}\hat{\gamma}_0\mathscr{L}^\dagger = \hat{\gamma}_0[(\bar{y}\cdot\bar{y})\cosh\phi - (\bar{x}\cdot\bar{x})\cos\theta] - J\hat{\gamma}_0\bar{x}\sin\theta\\ + J\hat{\gamma}_0\bar{y}\sinh\phi + J\hat{\gamma}_0\bar{x}\bar{y}(\cosh\phi - \cos\theta).\end{aligned} \tag{A2.31}$$

Similar calculations give us

$$\begin{aligned}\mathscr{L}(J\hat{\gamma}_0\bar{x}\bar{y})\mathscr{L}^\dagger = &-\hat{\gamma}_0(\bar{x}\cdot\bar{x})(\bar{y}\cdot\bar{y})[\cosh\phi - \cos\theta]\\ &+ J\hat{\gamma}_0\bar{x}(\bar{y}\cdot\bar{y})\sin\theta - J\hat{\gamma}_0\bar{y}(\bar{x}\cdot\bar{x})\sinh\phi\\ &- J\hat{\gamma}_0\bar{x}\bar{y}[(\bar{x}\cdot\bar{x})\cosh\phi - (\bar{y}\cdot\bar{y})\cos\theta],\end{aligned} \tag{A2.32}$$

$$\mathscr{L}(J\hat{\gamma}_0\bar{y})\mathscr{L}^\dagger = (\hat{\gamma}_0(\bar{y}\cdot\bar{y}) + J\hat{\gamma}_0\bar{x}\bar{y})\sinh\phi + J\hat{\gamma}_0\bar{y}\cosh\phi, \tag{A2.33}$$

and

$$\mathscr{L}(J\hat{\gamma}_0\bar{x})\mathscr{L}^\dagger = -[\hat{\gamma}_0(\bar{x}\cdot\bar{x}) + J\hat{\gamma}_0\bar{x}]\sin\theta + J\hat{\gamma}_0\bar{x}\cos\theta. \tag{A2.34}$$

If we multiply Eq. (A2.31) by $(\bar{y}\cdot\bar{y})$ and add the result to Eq. (A2.32) the trigonometric functions drop out. That is

$$\mathscr{L}[\hat{\gamma}_0(\bar{y}\cdot\bar{y}) + J\hat{\gamma}_0\bar{x}\bar{y}]\mathscr{L}^\dagger = (\hat{\gamma}_0(\bar{y}\cdot\bar{y}) + J\hat{\gamma}_0\bar{x}\bar{y})\cosh\phi + J\hat{\gamma}_0\bar{y}\sinh\phi. \tag{A2.35}$$

In a similar fashion, if we multiply Eq. (A2.31) by $(\bar{x}\cdot\bar{x})$ and add the result to Eq. (A2.32), the hyperbolic trigonometric functions drop out and we have:

$$\mathscr{L}[\hat{\gamma}_0(\bar{x}\cdot\bar{x}) + J\hat{\gamma}_0\bar{x}\bar{y}]\mathscr{L}^\dagger = (\hat{\gamma}_0(\bar{x}\cdot\bar{x}) + J\hat{\gamma}_0\bar{x}\bar{y})\cos\theta + J\hat{\gamma}_0\bar{x}\sin\theta. \tag{A2.36}$$

Adding and subtracting Eq. (A2.33) from Eq. (A2.35), gives us

$$\mathscr{L}[\hat{\gamma}_0(\bar{y}\cdot\bar{y}) + J\hat{\gamma}_0\bar{x}\bar{y} \pm J\hat{\gamma}_0\bar{y}]\mathscr{L}^\dagger = \mathrm{e}^{\pm\phi}[\hat{\gamma}_0(\bar{y}\cdot\bar{y}) + J\hat{\gamma}_0\bar{x}\bar{y} \pm J\hat{\gamma}_0\bar{y}].$$

From Eqs. (A2.36) and (A2.34), we also get

$$\mathscr{L}[\hat{\gamma}_0(\bar{x}\cdot\bar{x}) + J\hat{\gamma}_0\bar{x}\bar{y} \pm \mathrm{i}J\hat{\gamma}_0\bar{x}]\mathscr{L}^\dagger = \mathrm{e}^{\pm\mathrm{i}\theta}[\hat{\gamma}_0(\bar{x}\cdot\bar{x}) + J\hat{\gamma}_0\bar{x}\bar{y} \pm \mathrm{i}J\hat{\gamma}_0\bar{x}]. \quad \blacksquare$$

When $\mathscr{L} = I + b\hat{n}(I - J\hat{m})$ and $(\hat{n}\cdot\hat{m}) = 0$, the situation is quite different. This operator is described as a *null rotation*. Such operators for Lorentz spaces of arbitrary dimension were discussed in detail by Marcel Riesz (1993, pp. 84–127). For this 4-dimensional case, we have

Theorem A2.3. Suppose $\mathscr{L} = I + b\hat{n}(I - J\hat{m})$ where $(\hat{n}\cdot\hat{m}) = 0$. In this case there are only two eigenvectors each of which has an eigenvalue equal to 1. One is the space-like eigenvector $J\hat{\gamma}_0\hat{n}$. The other eigenvector is the light-like or null vector $\hat{\gamma}_0 + J\hat{\gamma}_0\hat{m}$. The null vector is part of a 3-dimensional subspace

spanned by $J\hat{\gamma}_0\hat{m}$, $J\hat{\gamma}_0\hat{m}\hat{n}$, and $\hat{\gamma}_0 + J\hat{\gamma}_0\hat{m}$ which is invariant under this transformation. Furthermore, suppose $(L - I)\vec{v} = \mathscr{L}\vec{v}\mathscr{L}^\dagger - \vec{v}$ where $\vec{v}$ is a 1-vector. Then the image of $L - I$ acting on the 3-dimensional space described above is the 2-dimensional space spanned by $J\hat{\gamma}_0\hat{m}\hat{n}$ and $\hat{\gamma}_0 + J\hat{\gamma}_0\hat{m}$. This 2-dimensional plane is described as light-like. It is tangent to the light cone and contains no time-like vector. In turn, the image of $L - I$ acting on this 2-dimensional space is the 1-dimensional space spanned by the light-like vector $\hat{\gamma}_0 + J\hat{\gamma}_0\hat{m}$. Finally the image of $L - I$ acting on this 1-dimensional space is 0.

Proof. To prove the theorem one simply shows that $\mathscr{L}(J\hat{\gamma}_0\hat{n})\mathscr{L}^\dagger = J\hat{\gamma}_0\hat{n}$,

$$\mathscr{L}(J\hat{\gamma}_0\hat{m})\mathscr{L}^\dagger = J\hat{\gamma}_0\hat{m} - 2bJ\hat{\gamma}_0\hat{m}\hat{n} - 2b^2(\hat{\gamma}_0 + J\hat{\gamma}_0\hat{m}),$$

$$\mathscr{L}(J\hat{\gamma}_0\hat{m}\hat{n})\mathscr{L}^\dagger = J\hat{\gamma}_0\hat{m}\hat{n} + 2b(\hat{\gamma}_0 + J\hat{\gamma}_0\hat{m}),$$

and

$$\mathscr{L}(\hat{\gamma}_0 + J\hat{\gamma}_0\hat{m})\mathscr{L}^\dagger = \hat{\gamma}_0 + J\hat{\gamma}_0\hat{m}.$$

This exercise is left to the reader. ■

From Eqs. (A2.34) and (A2.36), we see that when $\mathscr{L} = \exp(\frac{1}{2}(\phi - \theta J)\hat{z})$, $\mathscr{L}$ generates a rotation of angle θ in the 2-dimensional plane spanned by the two space-like vectors $\hat{\gamma}_0(\bar{x}\cdot\bar{x}) + J\hat{\gamma}_0\bar{x}\bar{y}$ and $J\hat{\gamma}_0\bar{x}$. (See Problem A2.1.)

From Eqs. (A2.33) and (A2.35), it is clear that the same Lorentz transformation generates a boost in the 2-dimensional plane spanned by the time-like vector $\hat{\gamma}_0(\bar{y}\cdot\bar{y}) + J\hat{\gamma}_0\bar{x}\bar{y}$ and the space-like vector $J\hat{\gamma}_0\bar{y}$. (See Problem A2.2.)

The eigenvectors $\hat{\gamma}_0(\bar{y}\cdot\bar{y}) + J\hat{\gamma}_0\bar{x}\bar{y} \pm J\hat{\gamma}_0\bar{y}$ are both *null* or *light-like* because they have length 0. Such vectors are important because they correspond to light rays.

Penrose and Rindler (1984, p. 28) make the observation that in general a restricted Lorentz transformation may be specified by taking any three distinct null vectors and then choosing the directions of their image null vectors. This implies that for a nontrivial Lorentz transformation, at most two null vector directions may be fixed. Penrose and Rindler go on to state that from topological arguments it can be shown that the direction of at least one null vector must be fixed.

The case where the directions of two null vectors are fixed corresponds to the eigenvectors $\hat{\gamma}_0(\bar{y}\cdot\bar{y}) + J\hat{\gamma}_0\bar{x}\bar{y} \pm J\hat{\gamma}_0\bar{y}$ for the Lorentz operator $\exp(\frac{1}{2}(\phi - \theta J)\hat{z})$.

The case where the direction of only one null vector is fixed corresponds to the null eigenvector $\hat{\gamma}_0 + J\hat{\gamma}_0\hat{m}$ for the null rotation $I + b\hat{n}(I - J\hat{m})$.

Problem A2.1. Show that the product of the vector $\hat{\gamma}_0(\bar{x}\cdot\bar{x}) + J\hat{\gamma}_0\bar{x}\bar{y}$ with itself is $-(\bar{x}\cdot\bar{x})$. This shows that the vector in question is space-like.

Problem A2.2. Show that the product of the vector $\hat{\gamma}_0(\bar{y}\cdot\bar{y}) + J\hat{\gamma}_0\bar{x}\bar{y}$ with itself is $(\bar{y}\cdot\bar{y})$. This shows that this vector is time-like.

Problem A2.3. According to Theorem A1.2, a restricted Lorentz operator $\mathscr{L}$ can be written as a product of the form $\mathscr{B}\mathscr{R}$ where $\mathscr{B}$ is a boost and $\mathscr{R}$ is a rotation. Compute $\mathscr{B}$ and $\mathscr{R}$ where $\mathscr{L}$ is the null rotation $I + b\hat{n}(I - J\hat{m})$. What is the angle between the axis of rotation and the direction of the velocity for the boost?

Problem A2.4. Suppose $P_+ = \frac{1}{2}(I + \hat{z})$, $P_- = \frac{1}{2}(I - \hat{z})$, $Q_+ = \frac{1}{2}(I + \mathrm{i}J\hat{z})$, and $Q_- = \frac{1}{2}(I - \mathrm{i}J\hat{z})$.

(1) Show P_+ and P_- are orthogonal *projection operators*, that is $(P_+)^2 = P_+$, $(P_-)^2 = P_-$, and $P_+P_- = P_-P_+ = 0$.
(2) Show Q_+ and Q_- satisfy a similar set of equations.
(3) Show $I = (P_+ + P_-)(Q_+ + Q_-) = P_+Q_+ + P_+Q_- + P_-Q_+ + P_-Q_-$.
(4) Show $\exp(\frac{1}{2}(\phi - \theta J)\hat{z})P_+Q_+ = \mathrm{e}^{\phi/2}\,\mathrm{e}^{\mathrm{i}\theta/2}P_+Q_+$.
(5) Show more generally that

$$\exp(\tfrac{1}{2}(\phi - \theta J)\hat{z})P_\pm Q_\pm = (\mathrm{e}^{\pm\phi/2}P_\pm)(\mathrm{e}^{\pm\mathrm{i}\theta/2}Q_\pm).$$

(6) Show

$$\exp(\tfrac{1}{2}(\phi - \theta J)\hat{z}) = \mathrm{e}^{(\phi+\mathrm{i}\theta)}P_+Q_+ + \mathrm{e}^{(\phi-\mathrm{i}\theta)}P_+Q_- + \mathrm{e}^{(-\phi+\mathrm{i}\theta)}P_-Q_+ + \mathrm{e}^{(-\phi-\mathrm{i}\theta)}P_-Q_-.$$

A.3 The Bianchi Identity

From Eq. (5.70):

$$(\nabla_j\nabla_k - \nabla_k\nabla_j)\mathscr{A} = -\tfrac{1}{2}\mathscr{R}_{jk}\mathscr{A} + \mathscr{A}\tfrac{1}{2}\mathscr{R}_{jk}. \tag{A3.1}$$

If we apply the operator ∇_i to both sides of this equation, we get

$$(\nabla_i\nabla_j\nabla_k - \nabla_i\nabla_k\nabla_j)\mathscr{A} = -\tfrac{1}{2}(\nabla_i\mathscr{R}_{jk})\,\mathscr{A} - \tfrac{1}{2}\mathscr{R}_{jk}\nabla_i\mathscr{A} + (\nabla_i\mathscr{A})\tfrac{1}{2}\mathscr{R}_{jk} + \mathscr{A}\tfrac{1}{2}\nabla_i\mathscr{R}_{jk}. \tag{A3.2}$$

If we now subject this equation to the permutation $i \to j \to k \to i$ twice, we get two more equations:

$$(\nabla_j\nabla_k\nabla_i - \nabla_j\nabla_i\nabla_k)\mathscr{A} = -\tfrac{1}{2}(\nabla_j\mathscr{R}_{ki})\mathscr{A} - \tfrac{1}{2}\mathscr{R}_{ki}\nabla_j\mathscr{A} + (\nabla_j\mathscr{A})\tfrac{1}{2}\mathscr{R}_{ki} + \mathscr{A}\tfrac{1}{2}\nabla_j\mathscr{R}_{ki} \tag{A3.3}$$

and

$$(\nabla_k \nabla_i \nabla_j - \nabla_k \nabla_j \nabla_i)\mathscr{A} = -\tfrac{1}{2}(\nabla_k \mathscr{R}_{ij})\mathscr{A} - \tfrac{1}{2}\mathscr{R}_{ij}\nabla_k \mathscr{A}$$
$$+ (\nabla_k \mathscr{A})\tfrac{1}{2}\mathscr{R}_{ij} + \mathscr{A}\tfrac{1}{2}\nabla_k \mathscr{R}_{ij} \qquad \text{(A3.4)}$$

Adding the last three equations together, we see that

$$6\nabla_{[i}\nabla_j\nabla_{k]}\mathscr{A} = -\tfrac{3}{2}\nabla_{[i}\mathscr{R}_{jk]}\mathscr{A} - \tfrac{3}{2}\mathscr{R}_{[ij}\nabla_{k]}\mathscr{A}$$
$$+ \tfrac{3}{2}(\nabla_{[i}\mathscr{A})\mathscr{R}_{jk]} + \tfrac{3}{2}\mathscr{A}\nabla_{[i}\mathscr{R}_{jk]}. \qquad \text{(A3.5)}$$

We now pursue the task of obtaining an alternative expression for the left-hand side of Eq. (A3.5). To achieve this, we first replace $\mathscr{A}$, by $\nabla_i \mathscr{A}$ in Eq. (A3.1) and then apply the same permutations that we did before. The three resulting equations are

$$(\nabla_j \nabla_k \nabla_i - \nabla_k \nabla_j \nabla_i)\mathscr{A} = -\tfrac{1}{2}\mathscr{R}_{jk}\nabla_i \mathscr{A} + (\nabla_i \mathscr{A})\tfrac{1}{2}\mathscr{R}_{jk},$$
$$(\nabla_k \nabla_i \nabla_j - \nabla_i \nabla_k \nabla_j)\mathscr{A} = -\tfrac{1}{2}\mathscr{R}_{ki}\nabla_j \mathscr{A} + (\nabla_j \mathscr{A})\tfrac{1}{2}\mathscr{R}_{ki},$$

and

$$(\nabla_i \nabla_j \nabla_k - \nabla_j \nabla_i \nabla_k)\mathscr{A} = -\tfrac{1}{2}\mathscr{R}_{ij}\nabla_k \mathscr{A} + (\nabla_k \mathscr{A})\tfrac{1}{2}\mathscr{R}_{ij},$$

Adding these last three equations now gives us

$$6\nabla_{[i}\nabla_j\nabla_{k]}\mathscr{A} = -\tfrac{3}{2}\mathscr{R}_{[ij}\nabla_{k]}\mathscr{A} + \tfrac{3}{2}(\nabla_{[i}\mathscr{A})\mathscr{R}_{jk]}. \qquad \text{(A3.6)}$$

By subtracting Eq. (A3.6) from Eq. (A3.5), we see that

$$0 = -\tfrac{3}{2}\nabla_{[i}\mathscr{R}_{jk]}\mathscr{A} + \tfrac{3}{2}\mathscr{A}\nabla_{[i}\mathscr{R}_{jk]}. \qquad \text{(A3.7)}$$

Since $\mathscr{A}$ is an arbitrary Clifford number, it is now clear that

$$\nabla_{[i}\mathscr{R}_{jk]} = 0. \qquad \text{(A3.8)}$$

Equation (A3.8) is the Bianchi identity. However, the usual form of the Bianchi identity is

$$R_{pq[ij;k]} = 0. \qquad \text{(A3.9)}$$

It is left to the reader to show that Eq. (A3.9) is equivalent to Eq. (A3.8).

BIBLIOGRAPHY

Abromowitz, Milton and Stegun, Irene A. (Editors) 1965. *Handbook of Mathematical Functions.* New York; Dover Publications, Inc.

Adler, Ronald; Bazin, Maurice; and Schiffer, Menachem 1975. *Introduction to General Relativity* 2nd Ed. New York; McGraw-Hill Book Company: Chapter 7.

Albert, Abraham Adrian 1939. *Structure of Algebras.* American Mathematical Society Colloquium Publications Vol. XXIV. Published by the American Mathematical Society 531 West 116th Street, New York City.

Atiyah, Michael F.; Bott, Raoul H.; and Shapiro, Arnold 1964. "Clifford modules." *Topology,* vol. 3. Suppl. 1, pp. 3–38.

Baym, Gordon 1969. *Lectures on Quantum Mechanics.* New York; W. A. Benjamin, Inc.

Bolker, Ethan D. 1973. "The spinor spanner." *The American Mathematical Monthly,* vol. 80, pp. 977–984.

Boyer, Robert H. and Lindquist, Richard W. 1967. "Maximal analytic extension of the Kerr metric." *Journal of Mathematical Physics,* vol. 8, No. 2, pp. 265–281.

Cartan, Élie 1938. *Leçons Sur La Théorie Des Spineurs I,* Exposés De Géométrie IX, Actualités Scientifiques Et Industrielles 643. Paris, France: Hermann et Cie, Éditeurs.

Cartan, Élie 1966. *The Theory of Spinors.* Cambridge, Massachusetts: The M.I.T. Press.

Chandrasekhar, Subrahmanyan 1983. *The Mathematical Theory of Black Holes.* New York: Oxford University Press.

Clifford, William Kingdon 1878. "Applications of Grassmann's extensive algebra." *American Journal of Mathematics* vol. 1, pp. 350–358. Republished 1882 in *Mathematical Papers by William Kingdon Clifford* (Edited by Robert Tucker) London: Macmillan and Co: pp. 266–276.

Clifford, William Kingdon 1882. "On the Classification of Geometric Algebras." Published posthumously in unfinished form as paper XLII in *Mathematical Papers by William Kingdon Clifford* (Edited by Robert Tucker) London: Macmillan and Co: pp. 397–401.

Cowan, Clyde L., Jr. and Reines, Frederick 1953. "The detection of the free neutrino." Letter in *Physical Review,* Second series, vol. 92, pp. 830–831.

Dieudonné, Jean 1948. *Sur Les Groupes Classiques,* Publications de l'Université de Strasbourg VI. Paris, France: Hermann et Cie, Éditeurs.

Dirac, Paul A. M. 1928. "The quantum theory of the electron." *Proceedings of the Royal Society of London* Series A, vol. 117, pp. 610–624.

Eddington, Arthur S. 1924. "A Comparison of Whitehead's and Einstein's formulas." *Nature*, vol. 113, p. 192.

Eddington, Arthur S. 1928. "A symmetrical treatment of the wave equation." *Proceedings of the Royal Society* (London) Series A. vol. 121, pp. 523–542.

Eddington, Arthur S. 1929. "The charge of an electron." *Proceedings of the Royal Society* (London) Series A, vol. 122, pp. 358–369.

Einstein, Albert 1905. "Zur electrodynamik bewegter körper." *Annalen der Physik*, Vierte folge, Band 17, (Leipzig), pp. 891–921.

Einstein, Albert 1915a. "Zur Allgemeinen Relativitätstheorie." *Sitzungsberichte der Preussischen Akademie der Wissenschaften zu Berlin*: pp. 778–786 and pp. 799–801.

Einstein, Albert 1915b. "Die Feldgleichungen der Gravitation." *Sitzungsberichte der Preussischen Akademie der Wissenschaften zu Berlin*: pp. 844–847.

Einstein, Albert 1915c. "Erklärung der Perihelbewegung des Merkur aus der allgemeinen Relativitätstheorie." *Sitzungsberichte der Preussischen Akademie der Wissenschaften zu Berlin*: pp. 831–839.

Einstein, Albert 1916. "Die Grundlage der allgemeinen Relativitätstheorie." *Annalen der Physik*, Vierte Folge, Band 49 (Leipzig), pp. 769–822.

Flanders, Harley 1963. *Differential Forms with Applications to the Physical Sciences.* New York: Academic Press.

Fock, Vladimir A. 1929. "Geometrisierung der Diracschen Theorie des Electrons." *Zeitschrift für Physik*, Band 57, pp. 261–277.

Fock, Vladimir A. and Ivanenko (Iwanenko), Dimitriĭ D. 1929. "Géométrie quantique linéaire et déplacement parallèle." *Comptes Rendus des Séances de L'Académie des Sciences*, Tome 188 (Paris): pp. 1470–1472.

Franz, Walter 1935. "Zur Methodik der Dirac-Gleichung." *Sitzungsberichte der mathematisch-naturwissenschaftlichen Abteilung der Bayenschen Akademie der Wissenschaften zu München*, Nr 3, pp. 379–435.

Georgi, Howard 1982. *Lie Algebras in Particle Physics.* Reading, Massachusetts: The Benjamin/Cummings Publishing Company, Inc.

Goldstein, Herbert 1980. *Classical Mechanics* 2nd Ed. Reading, Massachusetts: Addison-Wesley Publishing Company.

Gordon, Morton M. and Johnson, David A. 1974. "Basic orbit properties of ions in a migma fusion device." *Nuclear Instruments and Methods*, vol. 121, pp. 461–466.

Gordon, Walter 1928. "Der Strom der Diracschen Electronentheorie." *Zeitschrift für Physik*, Band 50 (Berlin), pp. 630–632.

Graham, Alexander 1981. *Kronecker Products and Matrix Calculus with Applications.* Chichester, West Sussex, England: Ellis Horwood Limited.

Guggenheimer, Heinrich W. 1977. *Differential Geometry.* New York: Dover Publications, Inc.

Hestenes, David 1966. *Space–Time Algebra.* New York: Gordon and Breach.

Hestenes, David and Sobczyk, Garret 1984. *Clifford Algebra to Geometric Calculus—A Unified Language for Mathematical Physics.* Dordrecht: D. Reidel Publishing Company.

Itzykson, Claude and Zuber, Jean-Bernard 1980. *Quantum Field Theory.* New York: McGraw-Hill Book Company.

Jackson, John David 1962. *Classical Electrodynamics.* New York: John Wiley and Sons, Inc.

Kaluza, Theodor 1921. "Zum Unitätsproblem der Physik." *Sitzungsberichte der Preussischen Akademie der Wissenschaften* (Berlin), pp. 966–972.

Kerr, Roy P. 1963. 'Gravitational field of a spinning mass as an example of algebraically special metrics." *Physical Review Letters,* vol. 11, Num. 5, pp. 237–238.

Kerr, Roy P. and Schild, Alfred 1965. "A new class of vacuum solutions of the Einstein field equations." *Proceedıngs of the Galileo Galilei Centenary Meeting on General Relativity, Problems of Energy and Gravitational Waves,* G. Barbera, ed., Comitato Nazionale per le Manifestazione Celebrative, Florence: pp. 222–233.

Kramer, D.; Stephani, H.; MacCallum, M.; and Herlt, E. 1980. *Exact Solutions of Einstein's Field Equations.* Cambridge: Cambridge University Press.

Lense, Von J. and Thirring, Hans 1918. "Über den Einfluss der Eigenrotation Zentralkörper auf die Bewegung der Planeten und Monde nach der Einsteinsch Gravitationstheorie." *Physikalische Zeitschrift* (Leipzig), vol. 19, pp. 156–163.

Lorentz, Hendrik A. 1904. "Electromagnetic phenomena in a system moving with any velocity smaller than light." *Proceedings of the Section of Sciences* Koninklijke Akademie van Wetenschappen te Amsterdam, vol. VI: pp. 809–831.

Lounesto, Pertti 1980. "Sur les idéaux à gauche algèbraes de Clifford et les produits scalaires des spineurs." *Annales de l'Institut Henri Poincaré* vol. *XXXIII* n° 1, Nouvelle Series *Section A* Physique Théorique, pp. 53–61.

Macek, Robert and Maglić (Maglich), Bogdan 1970. "The principle of self-colliding orbits and its possible application to π–π and μ–μ collisions." *Particle Accelerators,* vol. 1, pp. 121–136.

Maglić (Maglich), Bogdan C.; Blewett, John P.; Colleraine, Anthony P.; and Harrison, W. Craig 1971. "Fusion reactions in self-colliding orbits." *Physical Review Letters,* vol. 27, pp. 909–912.

Maglich, Bogdan C. 1973. "The migma principle of controlled fusion." *Nuclear Instruments and Methods,* vol. 111, pp. 213–235.

Magnus, Wilhelm; Oberhettinger, Fritz; and Soni, Raj Pal 1966. *Formulas and Theorems for the Special Functions of Mathematical Physics.* New York: Springer-Verlag.

Margenau, Henry and Murphy, George Moseley 1956. *The Mathematics of Physics and Chemistry* 2nd Ed. Princeton, New Jersey: D. Van Nostrand Company, Inc. p. 182.

Martin, Paul C. and Glauber, Roy J. 1958. "Relativistic theory of radiative orbital electron capture." *Physical Review,* vol. 109, No. 4, pp. 1307–1325.

Michelson, Albert A. 1881. "The relative motion of the Earth and the Luminiferous ether." *The American Journal of Science* Third series, vol. 22, pp. 120–129.

Michelson, Albert A. and Morley, Edward W. 1887. "On the relative motion of the Earth and the Luminiferous ether." *The American Journal of Science* Third Series, vol. 34, pp. 333–345.

Misner, Charles W. 1963. "The flatter regions of Newman, Unti, and Tanburino's generalized Schwarzschild space." *Journal of Mathematical Physics,* vol. 4, pp. 924–937.

Misner, Charles W.; Thorne, Kip S; and Wheeler, John Archibald 1973. *Gravitation.* San Francisco: W. H. Freeman and Company.

Newcomb, Simon 1898. "Tables of Mercury." *Astronomical Papers* 2nd Ed. vol. *VI*, prepared for the use of the *American Ephemeris and Nautical Almanac*: pp. 171–272.

Newman, Ezra and Penrose, Roger 1962. "An approach to gravitational radiation by a method of spin coefficients." *Journal of Mathematical Physics*, vol. 3, No 3 (New York), pp. 566–578.

Pauli, Wolfgang 1934. Instituts Solvay, Brussels. Institut International de physique, Conseil de physique. 7th, 1933. (Discussion following paper by Werner Heisenberg "La Structure du Noyau."): pp. 324–344.

Penrose, Roger and Rindler, Wolfgang 1984. *Spinors and Space–Time* vol. *1*. Cambridge: Cambridge University Press.

Penrose, Roger and Rindler, Wolfgang 1986. *Spinors and Space–Time vol. 2.* Cambridge: Cambridge University Press.

Petrov, Aleksei Zinoveivich 1954. "Classification of spaces defined by gravitational fields." Uch. zap. Kazan Gos. Univ. (Reports of the State University of Kazan, vol. 114, No. 8, p. 55.) English translation: Trans. No. 29, Jet Propulsion Lab. California Inst. Tech. Pasadena 1963.

Petrov, Aleksei Zinoveivich 1969. *Einstein Spaces* New York: Pergamon.

Poincaré, Henri 1905. "Sur la dynamique de l'électron." *Comptes Rendus* (Paris), vol. 140, pp. 1504–1508.

Porteous, Ian R. 1981. *Topological Geometry* 2nd Edition. Cambridge: Cambridge University Press.

Proca, Alexandre 1930. "Sur l'équation de Dirac." *Comptes Rendus* (Paris), vol. 190, pp. 1377–1379.

Riefin, Edgar 1979. "Some mechanisms related to Dirac's strings." *American Journal of Physics*, vol. 47, pp. 379–381.

Riesz, Marcel 1993. *Clifford Numbers and Spinors*, edited by E. Folke Bolinder and Pertti Lounesto. Dordrecht, The Netherlands: Kluwer Academic Publishers. Also available as Lecture series No. 38. 1958. College Park, Maryland: University of Maryland.

Sauter, Fritz 1930. "Lösung der Diracschen Gleichungen ohne Spezialisierung der Diracschen Operations." *Zeitschrift für Physik*, vol. 63, pp. 803–814.

Schwarzschild, Karl 1916. "Über das Gravitationsfeld eines Massenpunktes Nach der Einsteinschen Theorie." *Sitzber. Deut. Akad. Wiss. Berlin*, Kl. Math.-Phys. Tech.: pp. 189–196.

Sirlin, Alberto 1981. "A class of useful identities involving correlated direct products of γ matrices." *Nuclear Physics B* (Netherlands), vol. B192, pp. 93–99.

Sommerfeld, Arnold 1939. *Atombau und Spektrallinien* Band II. Braunschweig: Friedr. Vieweg und Sohn: pp. 217–341.

Stehney, Ann K. 1976. "Principal null directions without spinors." *Journal of Mathematical Physics*, vol. 17, No. 10, pp. 1793–1796.

Temple, George F. 1930. "The group properties of Dirac's operators." *Proceedings of the Royal Society* (London) Series A, vol. 127, pp. 339–348. Also: "The operational wave equation and the energy levels of the hydrogen atom." *Proceedings of the Royal Society* (London) Series A., vol. 127, pp. 349–360.

Wald, Robert M. 1984. *General Relativity*. Chicago: The University of Chicago Press.

Wehr, M. Russell; Richards, James A. Jr.; and Adair, Thomas W. III 1984. *Physics of the Atom* 4th Ed. Reading, Massachusetts: Addison-Wesley Publishing Company: pp. 180–184. Also 1978, 3rd Ed.: pp. 161–165.

INDEX